A Textbook of

Mathematics

Grade 11

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A Textbook of Mathematics
for Grade 11

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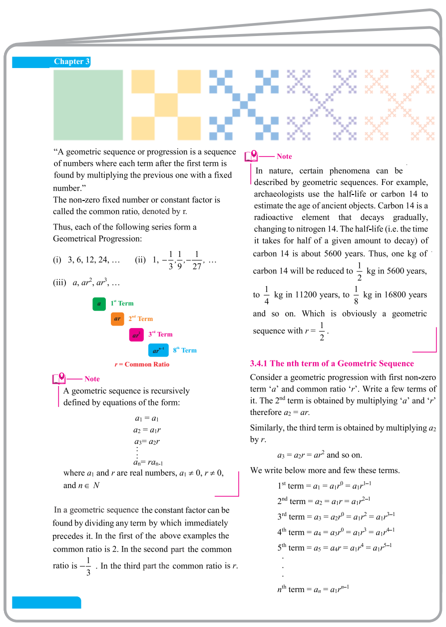
Preface

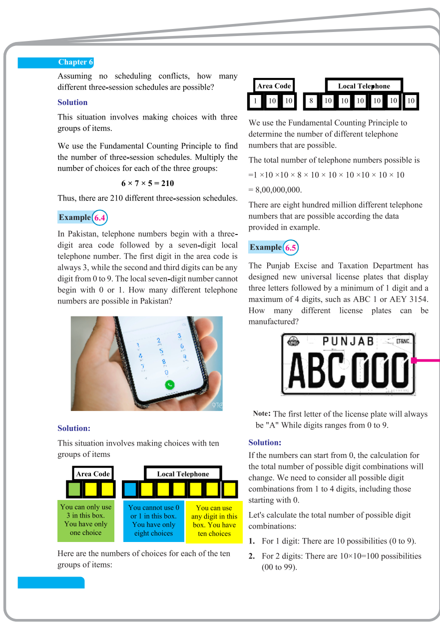
In this dynamic hybrid 11th-grade mathematics textbook, I embrace the evolving world of education by utilizing the CPA (Concrete, Pictorial, Abstract) Approach. This method starts with concrete examples that students can touch and see, moves to pictorial representations like diagrams and drawings, and finally to abstract concepts that use symbols and formulas. This step-by-step approach caters to diverse learning styles, making mathematics accessible and engaging for everyone.

The book covers essential topics such as Complex Numbers & Polar Form, Matrices and Determinants, Sequences and Series, Polynomial Division, Vectors in Space, Permutation and Combination, Mathematical Induction and Binomial Theorem, Fundamentals of Trigonometry, and Trigonometric Functions. Each chapter connects mathematical theories to real-life applications, transforming abstract concepts into vivid, relatable experiences.

Our textbook encourages active learning through "Test Yourself" sections, classroom activities, and "Teacher's Footnotes" to promote collaboration and critical thinking. Additionally, "Interesting Information" tabs provide useful insights and connections between the concepts and their real-world applications, enhancing understanding and relevance.

With a variety of examples, worksheets, video lectures, and simulations, we provide comprehensive practice and deepen understanding. This textbook is designed to instigate a deep appreciation for mathematics, focusing on practical applications and helping students see the relevance of math in everyday life. It also serves as the best demonstration of SLO (Student Learning Outcomes) based exams, catering to all requirements and needs for such assessments. It's more than an educational tool; it's a journey into the beauty and utility of mathematics in the modern world.





The purpose of a skill is to apply knowledge. Students and teachers can scan the provided QR code to access a worksheet that enhances their understanding.

How to use this book

The CPA approach, enhanced with interactive images, makes mathematics more accessible and engaging, transforming abstract concepts into tangible visuals for deeper understanding and effective learning.

Knowledge is information about a specific topic that helps clarify concepts. Students and teachers can scan the QR code provided with the SLO Tab to access lectures related to that topic.

Detailed **Solutions** will be available in QR Code given at the start of Review Exercise of Every Chapter

SLO based Model Video lecture

Salient Features

Comprehensive Learning
Engage students with videos, simulations, and practical worksheets.

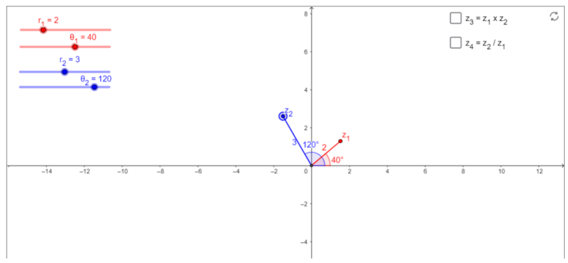
Structured Lesson Plan
Well-organized with clear objectives, PPTs, and a question bank.

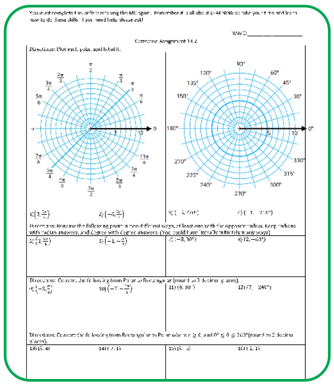
Engaging Multimedia
Visual appeal through PPTs and interactive simulations.

Assessment & Tracking
Diverse question bank and progress monitoring.

Adaptable & Accessible
Scalable and accessible, suitable for all learners.

Simulation





Skills Sheet

SLO No: M - 11 - A - 12
Apply the operations with complex numbers in polar form

$x = \tan^{-1}\frac{4}{3} = 0.927\,(to\ 3\ decimals)$

So $3+4i$ can also be $5e^{0.927i}$

Example 1.18 Write $3e^{\frac{\pi}{6}i}$ in complex $a+bi$ form

Solution:

$3e^{\frac{\pi}{6}i} = 3\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$= 3\left[\frac{\sqrt{3}}{2} + i.\frac{1}{2}\right]$

$= \frac{3\sqrt{3}}{2} + i\frac{1}{2}$

Example 1.19 Find the polar form of $-4+4i$

Solution:

On the complex plane, this complex number would correspond to the point $(-4,4)$ on a Cartesian plane.

We Can find the distance r and angle θ as we did in the last section

$r^2 = x^2 + y^2\ r^2 = (-4)^2 + (4)^2\ r = \sqrt{32} = 4\sqrt{2}$

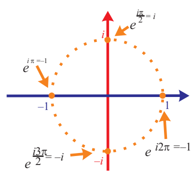
To find θ we can use $\theta = \pi - \tan^{-1}\left|\frac{4}{4}\right| = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

as θ lies in 2nd Quadrant

$r = 4\sqrt{2},\ \theta = \frac{3\pi}{4}\quad z = re^{i\theta} = 4\sqrt{2}e^{\frac{3\pi}{4}i}$

Interesting Information

Euler's theorem is crucial in securing online transactions. It ensures that sensitive data, like credit card information, can be encrypted and decrypted securely between the buyer and the online store.



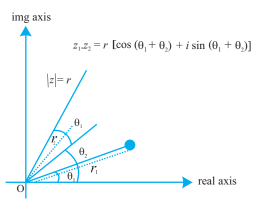
1.3.4 Products Of Complex Numbers In Polar Form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$z_1z_2 = r_1r_2 cis(\theta_1+\theta_2)$



Notice that the product calls for multiplying the moduli and adding the angles.

Example 1.20

Multiply $4(\cos 30^\circ + i\sin 30^\circ)$ by $2(\cos 60^\circ + i\sin 60^\circ)$

Solution:

Let $z_1 = 4(\cos 30^\circ + i\sin 30^\circ)$

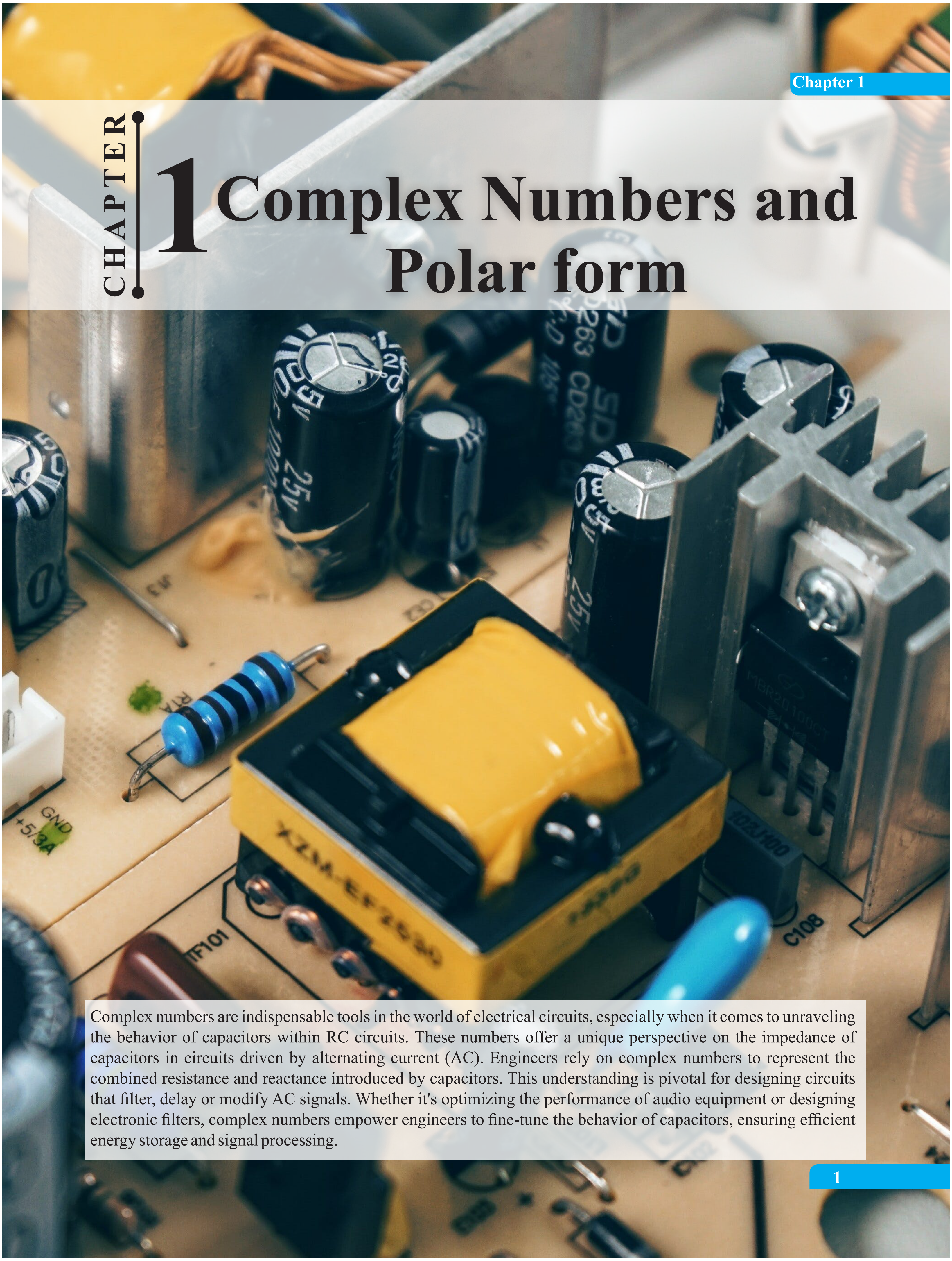
and $z_2 = 2(\cos 60^\circ + i\sin 60^\circ)$

Then

$z_1.z_2 = 4\times 2[\cos(30^\circ+60^\circ) + i\sin(30^\circ+60^\circ)]$

$z_1.z_2 = 8(\cos 90^\circ + i\sin 90^\circ) = 8cis90^\circ$

Content List



CHAPTER 1 Complex Numbers and Polar form

Complex numbers are indispensable tools in the world of electrical circuits, especially when it comes to unraveling the behavior of capacitors within RC circuits. These numbers offer a unique perspective on the impedance of capacitors in circuits driven by alternating current (AC). Engineers rely on complex numbers to represent the combined resistance and reactance introduced by capacitors. This understanding is pivotal for designing circuits that filter, delay or modify AC signals. Whether it's optimizing the performance of audio equipment or designing electronic filters, complex numbers empower engineers to fine-tune the behavior of capacitors, ensuring efficient energy storage and signal processing.

Student Learning Outcomes

1. Recall complex number $z = a + ib$ represented by an expression of the form (a, b) or of the form $i = \sqrt{-1}$
2. Recognize a as a real part of z and b as an imaginary part of z.
3. Know the condition for equality of complex numbers.
4. Carry out basic operations on complex numbers
5. Define $z = a - ib$ as the complex conjugate of $z = a + ib$
6. Define $|z| = \sqrt{a^2 + b^2}$ as the absolute value or modulus of a complex number $z = a+ib$
7. Solve the simultaneous linear equations with complex coefficients. For example, $5z - (3+i)w = 7 - i$; $(2-i)z + 2iw = -1 + i$
8. Write the polynomial $p(z)$ as a product of linear factors. For example, $z^2 + a^2 = (z + ia)(z - ia)$; $z^3 - 3z^2 + z + 5 = (z+1)(z-2-i)(z-2+i)$
9. Solve quadratic equation of the form $2pz + qz + r$ by completing squares, where p, q, r are real numbers and z a complex number. For example, solve: $z^2 - 2z + 5 = 0$; $(z-1-2i)(z-1+2i) = 0$; $z = 1 + 2i, 1 - 2i$
10. Explain the polar coordinates system.
11. Describe the polar representation of a complex number.
12. Apply the operations with complex numbers in polar representation.
13. Demonstrate simple equations and in-equations involving complex numbers in polar form.
14. Apply concepts of complex numbers to real world problems (such as cryptography, wave phenomena, calculate voltage, current, circuits, the velocity and pressure of the fluid).

Knowledge

1. **Representation of Complex Numbers:**
   z represented as $a + ib$ or (a,b), where a and b are real numbers and $i = \sqrt{-1}$.
2. **Understanding Real and Imaginary Parts:**
   Recognition of a as the real part and b as the imaginary part of z.
3. **Condition for Equality:**
   Knowledge of the condition for the equality of complex numbers.
4. **Complex Conjugate:**
   Definition of the complex conjugate: $z = a - ib$ as the complex conjugate of $z = a + ib$.
5. **Absolute Value or Modulus:**
   Definition of the absolute value or modulus of a complex number: $|z| = \sqrt{a^2 + b^2}$
6. **Solving Simultaneous Linear Equations:**
   Ability to solve simultaneous linear equations with complex coefficients.
7. **Factoring Polynomials:**
   Writing polynomials as a product of linear factors, e.g., $z^2 + a^2 = (z + ia)(z - ia)$ or
   $z^3 - 3z^2 + z + 5 = (z + 1)(z - 2 - i)(z - 2 + I)$
8. **Solving Quadratic Equations:**
   Solving quadratic equations of the form $pz^2 + qz + r = 0$ by completing squares, with p, q, r as real numbers and z as a complex number.
9. **Understanding Polar Coordinates:**
   Explanation of the polar coordinate system.
10. **Polar Representation of Complex Numbers:**
    Description of the polar representation of a complex number.
11. **Operations in Polar Form:**
    Application of operations (addition, subtraction, multiplication, division) with complex numbers in polar representation.
12. **Application to Equations:** Demonstration of equations involving complex numbers in polar form.
13. **Real-World Application:** Application of complex number concepts to real-world problems, such as cryptography, wave phenomena, electrical circuits, fluid dynamics, voltage, current, velocity, pressure, etc.

Pre & Post Requisite

Class 9
Chapter # 1
Real Number System

Class 10
Chapter # 1
Complex Numbers

Class 11
Chapter # 1
Complex Numbers

Skills

1. **Understanding Complex Number Representation:**
   - Ability to recall and understand the representation of a complex number z as $z = a+ib$, where a and b are real numbers and $i = \sqrt{-1}$
2. **Recognition of Real and Imaginary Parts:**
   - Proficiency in identifying and differentiating a as the real part and b as the imaginary part of a complex number z.
3. **Knowledge of Equality Conditions:**
   - Understanding the conditions necessary for the equality of complex numbers.
4. **Performing Basic Operations on Complex Numbers:**
   - Skill in carrying out fundamental mathematical operations (addition, subtraction, multiplication, division) involving complex numbers.
5. **Defining Complex Conjugate:**
   - Ability to define the complex conjugates $(\bar{z} = a - ib)$ as the conjugate of a complex number $(z = a + ib)$.
6. **Understanding Absolute Value or Modulus:**
   - Skill in defining and calculating the absolute value or modulus $(|z| = \sqrt{a^2 + b^2})$ of a complex number z.
7. **Solving Simultaneous Linear Equations with Complex Coefficients:**
   - Proficiency in solving sets of simultaneous linear equations involving complex coefficients.
8. **Factoring Polynomials into Linear Factors:**
   - Skill in expressing polynomials as products of linear factors, enabling the representation of polynomial equations in simpler forms.
9. **Solving Quadratic Equations using Completing the Square Method:**
   - Ability to solve quadratic equations of the form $pz^2 + qz + r = 0$ by completing squares, where p, q, r are real numbers and z is a complex number.
10. **Understanding Polar Representation:**
    - Interpret and demonstrate comprehension of the polar representation of complex numbers.
    - Applying operations in polar form.
    - Apply operations $(+, -, \times, \div)$ with complex numbers proficiently in polar representation.
11. **Application to Equations:**
    - Demonstration of equations & inequations involving complex numbers in polar form.
12. **Applying Concepts in Real-World Contexts:**
    - Apply complex number concepts practically to real-world scenarios such as cryptography, wave phenomena, electrical circuits, fluid dynamics, etc

Introduction

Throughout our studies in mathematics and science, you've encountered numerous concepts and problems where real numbers were sufficient to find solutions. However, as we dive deeper into mathematics and its applications in fields like physics, aeronautical engineering, and electrical engineering, a new set of numbers called "complex numbers" becomes essential.

In real numbers the square of any number 'n' is always positive. i.e, $n^2 \geq 0$.

Now, look at the following examples.

$n^2 + 1 = 0$, $n^2 + 9 = 0$, $n^2 + 10 = 0$

Consider the equation $n^2 + 1 = 0$

$$\boxed{n^2 = -1} \longrightarrow \text{(i)}$$

has no solution among the real numbers.

That's why is we cannot find a real number which satisfies equation (i).

Similarly, $n^2 + 9 = 0$

or $n^2 = -9 \longrightarrow$ (ii)

has no solution and so on.

To resolve this issue, mathematicians introduced a new number denoted by a Greek letter of alphabet ***i*** (iota) such that $i = \sqrt{-1}$ which is called the **imaginary number**. If we square both sides of the equation, we have $i^2 = -1$, a result that can never be obtained with the real numbers. So, by the definition.

$$i = \sqrt{-1}\ ,\ i^2 = -1$$

Using the above definition, we now have solutions for the equations (i) and (ii) as follows:

$$n^2 = -1$$

$n = \pm\sqrt{-1} = \pm i$ and $n^2 = -9$, $n = \pm\sqrt{-9}$

$$= \pm\sqrt{-1 \times 9} = \pm 3i$$

A simple consequence of the definition of i is that all powers of i may be expressed in terms of ± 1 and i itself.

For example, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = \left(i^2\right)^2 = 1$ and if we continue in this way to obtain higher powers of i, we obtain the values $1, i, -1$ or $-i$.

Patterns of i			
			Exponents
i	$\sqrt{-1}$	i	5 9 13 17 21 25
i^2	$\sqrt{-1} \times \sqrt{-1}$	-1	6 10 14 18 22 26
i^3	$(-1)(\sqrt{-1})$	$-i$	7 11 15 19 23 27
i^4	$(-1)(-1)$	1	8 12 16 20 24 28

Discovery

Once upon a time, imaginary numbers were just a wild idea that many smart people thought was just too strange to be true. René Descartes even called them "imaginary" to show how odd he thought they were. But then, brilliant minds like Euler and Gauss showed everyone how these unusual numbers could solve problems no one could solve before. This turned imaginary numbers from a curious idea into a superstar in math, changing the game in many fields!

Student Learning Outcomes

- Recall complex number z represented by an expression of the form $z = a + ib$ or of the form (a, b) where a and b are real numbers and $i = \sqrt{-1}$
- Recognize a as a real part of z and b as an imaginary part of z.
- Know the condition for equality of complex numbers
- Carryout basic operations on complex numbers
- Define $z = a - ib$ as the complex conjugate of $z = a + ib$
- Define $|z| = \sqrt{a^2 + b^2}$ as the absolute value or modulus of a complex number $z = a + ib$

1.1 Complex Numbers

***z* represented by an expression of the form $z = a + ib$**

Since, the complex number is any number that can be written as $a+bi$ where a and b are real numbers.

The real number a is called the **real part** of and the real number b is called the **imaginary part** of $a+bi$.

For example, the complex number $-4+5i$ has the real part $a=-4$ and the imaginary part $b=5$.

Usually, the complex number $a+bi$ is denoted by z. Accordingly,

$$z_1 = a_1 + b_1 i,\ \ z_2 = a_2 + b_2 i, z_3 = a_3 + b_3 i, \ldots$$

The set of all complex numbers is denoted by C, that is $C = \{a+bi \mid a,b \text{ are real numbers and } \sqrt{-1} = i\}$

1.1.1 Complex Numbers as Ordered Pairs of Real Numbers (*a* as real part and *b* as imaginary part of *z*)

Complex numbers may also be defined as ordered pairs of real numbers. Thus, a **complex number** z is an ordered pair (a,b) of real numbers a and b, written as $\boldsymbol{z=(a,b)}$

The first component a is called the **real part** of z and the second component b is called the **imaginary part** The real part is denoted by Re(z) and imaginary part is denoted by $Im(z)$ respectively.

For Example $Re(z)=a$ and $Im(z)=b$. The ordered pair (0,1) only has the **imaginary part** and it is denoted by $\boldsymbol{i}=(0,1)$.

The set of all ordered pairs of real numbers is the set of complex numbers denoted by C, that is $C = \{(a,b) \mid a,b \text{ are real numbers}\} = \mathbb{R}\times\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers.

Note

In $x + yi$, if y = 0, then $x + yi = x + 0i = x$ is a real number. Thus, every real number x can be written as a complex number by choosing $y = 0$. If $x = 0$ and $y \neq 0$, then $x + yi = 0 + yi = yi$ is known as **pure imaginary number.**

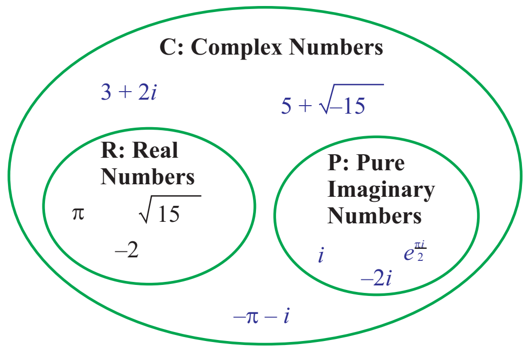


Figure 1.1
Illustration of real numbers as a subset of complex numbers

1.1.2 Operations on Complex Numbers in Ordered Pair Form

The equality and operations of addition, subtraction, multiplication and division in the set of complex numbers C are defined as follows.

(i) Equality:

$$(a_1,b_1)=(a_2,b_2) \Leftrightarrow \quad a_1=a_2, b_1=b_2$$

(ii) Addition:

$$(a_1,b_2)+(a_2,b_2)=(a_1+a_2,b_1+b_2)$$

(iii) Subtraction:

$$(a_1,b_1)-(a_2,b_2)=(a_1-a_2,b_1-b_2)$$

(iv) Multiplication:

$$(a_1,b_1)(a_2,b_2)=(a_1a_2-b_1b_2,a_1b_2+b_1a_2)$$

(v) Scalar Multiplication:

$k\,(a, b) = (ka, kb)$ for any real number k.

(vi) Division:

$$\frac{(a_1,b_1)}{(a_2,b_2)} = \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}, \frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}\right), (a_2,b_2)\neq(0,0)$$

The set of all complex numbers (a, b) in which the second component is zero has all the properties of set of "real numbers". For example, addition and multiplication of $(a_1,0)$ and $(a_2,0)$ give

$(a_1,0)+(a_2,0)=(a_1+a_2,0)$ by (ii)

$(a_1,0)(a_2,0)=(a_1a_2,0)$ by (iv)

which are numbers of the same type with imaginary part equal to zero. So, we can write $a=(a, 0)$.

Now considering, $i=(0, 1)$,

we have

$$i^2=i.i=(0,1)(0,1)=(-1,0)=-1 \quad \because a=(a, 0)$$

that is, $i^2=-1$. Now, we are in a position to express every complex number z as an ordered pair in terms of i as follows;

$z=(a,b)=(a,0)+(0,b)$ by (ii)

$=(a,0)+(b,0)(0,1)$ by (iv)

$=a+bi$ ($\because a=(a, 0)$ and $i=(0, 1)$)

that is $z=(a,b)=a+bi$

We see that an ordered pair (a,b) is expressible in the usual form of complex number as $a+bi$. Thus the two notations for a complex numbers z can be used interchangeably.

Example 1.1 Write the following in form of ordered pair.

(i) 7	**(ii)** $3i$	**(iii)** 0
(iv) $\frac{1}{2}$	**(v)** $3-\sqrt{-16}$	**(vi)** 1.

Solution:

(i) $7=7+0i=(7,0)$ **(ii)** $3i=0+3i=(0,3)$

(iii) $0=0+0i=(0,0)$

(iv) $\frac{1}{2}=\frac{1}{2}+0i=\left(\frac{1}{2},0\right)$

(v) $3-\sqrt{-16}=3-i\sqrt{16}=3-4i=(3,-4)$

(vi) $1=1+0i=(1, 0)$

- In (iii) of example 1.1, we see that 0 can be expressed as a sum of real and imaginary number and hence is a complex number. Such a complex number whose real and imaginary parts are zero is called **zero complex number**.
- Similarly, in (vi) of example 1.1, 1 can be expressed as a complex number with real part 1 and imaginary part 0. The complex number 1 is known as the **unity in complex number**.

1.1.3 Condition for equality of Complex Numbers:

If $z_1=a_1+b_1i$ and $z_2=a_2+b_2i$ are two complex numbers, then

$a_1+b_1i=a_2+b_2i$

$\Leftrightarrow (a_1+b_1i)-(a_2+b_2i)=0$

$\Leftrightarrow (a_1-a_2)+(b_1-b_2)i=0$

$\Leftrightarrow (a_1-a_2)=0$ and $(b_1-b_2)=0$

$\Leftrightarrow a_1=a_2$ and $b_1=b_2$

that is $\boxed{a_1+b_1i=a_2+b_2i \Leftrightarrow a_1=a_2 \text{ and } b_1=b_2}$

"Two Complex numbers are said to be equal if their real and imaginary parts are equal to each other"

Example 1.2 If a, b are real numbers and $7a+i(3a-b)=14-6i$, then find the values of a and b.

Solution:

Given, $7a+i(3a-b)=14-6i$

$\Rightarrow 7a+i(3a-b)=14+i(-6)$

Now equating real and imaginary parts on both sides, we have

$7a=14$ and $3a-b=-6$

$\Rightarrow a=2$ and $3(2)-b=-6$

$\Rightarrow a=2$ and $6-b=-6$

$\Rightarrow a=2$ and $-b=-12$

$\Rightarrow a=2$ and $b=12$

Therefore, the value of $a=2$ and the value of $b=12$.

Example 1.3 For what real values of m and n are the complex numbers $m^2-7m+9ni$ and $n^2i+20i-12$ are equal.

Solution:

Given complex numbers are $m^2-7m+9ni$ and $n^2i+20i-12$

According to the problem,

$$m^2-7m+9ni=n^2i+20i-12$$

$$\Rightarrow \left(m^2-7m\right)+i\left(9n\right)=\left(-12\right)+i\left(n^2+20\right)$$

Now equating real and imaginary parts on both sides, we have

$$m^2-7m=-12 \text{ and } 9n=n^2+20$$

$$\Rightarrow m^2-7m+12=0 \text{ and } n^2-9n+20=0$$

$$\Rightarrow (m-4)(m-3)=0 \text{ and } (n-5)(n-4)=0$$

$$\Rightarrow m=4,3 \text{ and } n=5,4$$

Hence, the required values of m and n are follows:

($m = 4$, $n = 5$) ; ($m = 4$, $n = 4$) ; ($m = 3$, $n = 5$) ; ($m = 3$, $n = 4$).

1.1.4 Basic Algebraic Operations on Complex Numbers

Let $z_1 = a_1 + b_1$i and $z_2 = c + b_2$i be two complex numbers. Then their,

(i) Addition:

$$\boxed{z_1+z_2=\left(a_1+b_1i\right)+\left(a_2+b_2i\right)=\left(a_1+b_2\right)+\left(b_1+b_2\right)i}$$

(ii) Subtraction:

$$\boxed{z_1-z_2=\left(a_1-b_1i\right)-\left(a_2-b_2i\right)=\left(a_1-a_2\right)-\left(b_1-b_2\right)i}$$

Example 1.4 Perform the indicated operation in each of the following.

(i) $(8-5i)+(5+6i)$ **(ii)** $i-(6-9i)$.

Solution:

(i) $(8-5i)+(5+6i)=(8+5)+(-5+6)i=13+i$

(ii) $i-(6-9i)=(0-6)+(1-(-9))i=-6+10i$

(iii) Multiplication:

$$z_1z_2 = (a_1+b_1i)(a_2+b_2i)$$
$$= a_1(a_2+b_2i)+b_1i(a_2+b_2i)$$
$$= a_1a_2+a_1b_2i+b_1a_2i+b_1b_2i^2$$
$$= a_1a_2+a_1b_2i+b_1a_2i-b_1b_2$$
$$= (a_1a_2-b_1b_2)+(a_1b_2+b_1a_2)i$$

that is

$$\boxed{z_1z_2=\left(a_1+b_1i\right)\left(a_2+b_2i\right)=\left(a_1a_2-b_1b_2\right)+\left(a_1b_2+b_1a_2\right)i}$$

and for scalar multiplication

$$\boxed{kz_1=k\left(a_1+b_1i\right)=ka_1+kb_1i \text{ for any real number } k.}$$

Example 1.5 Multiply $(2+3i)(4+7i)$.

Solution:

$$(2+3i)(4+7i)=(2)(4)+(2)(7i)+(4)(3i)+(3i)(7i)$$
$$=8+14i+12i+21(-1)$$
$$=(8-21)+(14+12)i$$
$$=-13+26i.$$

Note

When performing operations with square roots of negative numbers, begin by expressing all square roots in terms of i. Then perform the indicated operation.

Incorrect:

$$\cancel{\sqrt{-25}.\sqrt{-4}=\sqrt{(-25)(-4)}}$$
$$\cancel{\sqrt{100}}$$

Correct:

$$\sqrt{-25}.\sqrt{-4}=i\sqrt{25}\times i\sqrt{4}=5i\times 2i=10i^2$$
$$=10(-1)=-10$$

Division

(iv) Division of Complex Numbers:

The division of one complex number by another complex number cannot be carried out, because the denominator consists of two independent terms. This difficulty can be overcome by multiplying the numerator and denominator by the conjugate of the complex number in the denominator. This process is known as **rationalization**.

We have $\frac{z_1}{z_2}=\frac{a_1+b_1i}{a_2+b_2i}$

$$=\frac{a_1+b_1i}{a_2+b_2i}\times\frac{a_2-b_2i}{a_2-b_2i} \quad \text{(by rationalization)}$$

$$=\frac{(a_1+b_1i)}{(a_2+b_2i)}\times\frac{(a_2-b_2i)}{(a_2-b_2i)}$$

$$=\frac{(a_1a_2+b_1b_2)-(a_1b_2-b_1a_2)i}{a_2^2+b_2^2}$$

$$=\frac{(a_1a_1+b_1b_2)-(a_1b_2-b_1a_2)i}{a_2^2+b_2^2}$$

$$=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}i$$

Thus $\boxed{\frac{z_1}{z_2}=\frac{a_1+b_1i}{a_2+b_2i}=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}i}$

Example 1.6 Write $\frac{2+3i}{3-5i}$ in the form $a + bi$.

Solution: $\frac{2+3i}{3-5i}\times\frac{3+5i}{3+5i}$

$$=\frac{(2+3i)(3+5i)}{(3-5i)(3+5i)}$$

$$=\frac{6+10i+9i+15(i)^2}{9+15i-15i-25(i)^2}$$

$$=\frac{6+19i+15(-1)}{9-25(-1)}$$

$$=\frac{-9+19i}{34}=-\frac{9}{34}+\frac{19}{34}i$$

1.1.5 Conjugate of a Complex Number

The conjugate of the complex number $a+bi$ is $a-bi$ and $a-bi$ is $a+bi$. We denote the conjugate of any complex number z as $\bar{z}$ is obtained by changing the sign of the imaginary part of z.

Thus, Conjugate of $z=a+bi$ is $\bar{z}=a-bi$

Multiplying $a + bi$ by its conjugate we find

$(a+bi)(a-bi)=a^2+b^2+0i=a^2+b^2$.

Thus, a complex number times its conjugate is always real. i.e., its imaginary part is zero.

Example 1.7 Find the conjugate of **(i)** $-4-5i$ and **(ii)** $6+9i$.

Solution: **(i)** $z=-4-5i$

$\bar{z}=-4+5i$

(ii) $z=6+9i$

$\bar{z}=6-9i$

1.1.6 Graphical Representation of Complex Numbers

We can represent complex numbers in the complex plane. For this purpose, we use horizontal axis and vertical axis. Every point in the plane may be associated with just one complex number. Thus, there is (1–1) correspondence between the infinite set of complex numbers and the points of the plane.

In this representation of z, the real part of z is taken along x-axis of the plane and the imaginary part of z is taken along y-axis of the plane (figure 1.2). The x-axis and y-axis are referred as to **real axis** and **imaginary axis** respectively.

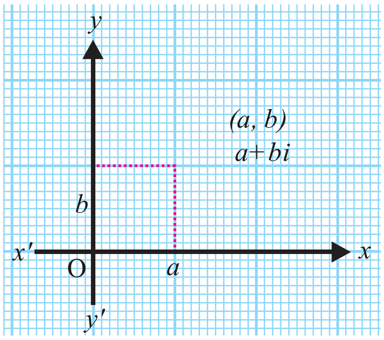


Figure 1.2

Example 1.8 Represent the following complex numbers on the complex plane.

(a) $-2+3i$ **(b)** $4-5i$ **(c)** $-2+3i$ **(d)** $-8-2i$

Solution: In figure 1.3, all the above complex numbers have been represented. We see that the complex numbers appear in all the four quadrants due to the negative and positive signs with their real and imaginary parts.

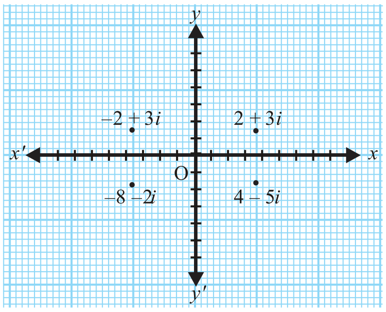


Figure 1.3

1.1.7 Absolute value or Modulus of a Complex Number

Let $z = (a, b) = a + bi$ be a complex number. Then **absolute value** or **modulus** of z, denoted by $|z|$, is defined by

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

In the figure 1.4 ,Point P represents $a + bi$. $\overline{PQ}$ is a perpendicular drawn on x-axis.

Thus, $\overline{OQ} = a$ and $\overline{PQ} = b$. In the right angled–triangle OQP, we have, by Pythagoras theorem

$$|\overline{OP}|^2 = |\overline{OQ}|^2 + |\overline{PQ}|^2 = a^2 + b^2$$

$$\therefore \quad |\overline{OP}| = \sqrt{a^2 + b^2} = |z|$$

Therefore, the modulus of a complex number is the distance from the origin of the point representing the number.

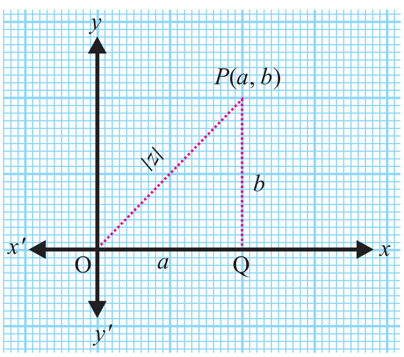


Figure 1.4

Example 1.9

Compute the absolute value of the following complex numbers:

(i) $2i$ **(ii)** 4 **(iii)** $3 - 6i$

Solution:

(i) Let $z = 2i$ or $z = 0 + 2i$

Then by the definition

$$|z| = \sqrt{(0)^2 + (2)^2} = \sqrt{2^2} = 2$$

(ii) Let $z = 4$ or $z = 4 + 0i$.

Then by the definition

$$|z| = \sqrt{(4)^2 + (0)^2} = \sqrt{4^2} = 4$$

(iii) Let $z = 3 - 6i$.

Then by the definition

$$|z| = \sqrt{(3)^2 + (-6)^2} = \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$$

Interesting Information

Did you know that in communications, complex numbers help make sure our phone calls and internet work well? When signals are sent through the air or wires, they can be thought of as complex numbers. The absolute value of these numbers tells us how strong the signals are. By checking this strength, engineers can figure out if the signals are clear or if there's a problem. This helps make sure that when you call a friend or watch a video online, the sound and picture come through smoothly without interruptions.

Skill 1.1

Understanding Complex Number Representation:
Ability to recall and understand the representation of a complex number z as $z = a+ib$, where a and b are real numbers and $i = \sqrt{-1}$

Recognition of Real and Imaginary Parts:
Proficiency in identifying and differentiating a as the real part and b as the imaginary part of a complex number z.

Knowledge of Equality Conditions:
Understanding the conditions necessary for the equality of complex numbers.

Performing Basic Operations on Complex Numbers:
Skill in carrying out fundamental mathematical operations (addition, subtraction, multiplication, division) involving complex numbers.

Defining Complex Conjugate:
Ability to define the complex conjugates $(z = a - ib)$ as the conjugate of a complex number $(z = a + ib)$.

Understanding Absolute Value or Modulus:
Skill in defining and calculating the absolute value or modulus ($|z| = \sqrt{a^2 + b^2}$) of a complex number z.

Exercise 1.1

1. Simplify and write the complex number as $i, -i, -1$ and 1

(i) $-i^{40}$ **(ii)** i^{223} **(iii)** i^{-21} **(iv)** i^{0} **(v)** i^{-1}

2. Add the following complex numbers.

(i) $4(2+3i), -3(1-2i)$ **(ii)** $\frac{1}{3}-\frac{2}{3}i, \frac{1}{2}-\frac{1}{4}i$ **(iii)** $(\sqrt{3},1), (1,\sqrt{3})$ **(iv)** $\left(\frac{4}{5},\frac{\sqrt{3}}{4}\right), \left(\frac{\sqrt{3}}{4},\frac{4}{5}\right)$

3. Subtract the following complex numbers.

(i) $2\sqrt{2}-5\sqrt{7}i$ from $5\sqrt{2}-9\sqrt{7}i$ **(ii)** $\left(-7,\frac{1}{3}\right)$ from $\left(7,\frac{1}{3}\right)$

(iii) $(x,0)$ from $(3,-y)$ **(iv)** $2x-3yi$ from $4x-7yi$

4. Multiply the following complex numbers:

(i) $(8i+11)(-7+5i)$ **(ii)** $(5i)(1-2i)$ **(iii)** $(9-12i)(15i+7)$

5. Perform the division and write the answer in the form $a + bi$.

(i) $\frac{4+i}{3+5i}$ **(ii)** $\frac{1}{-8+i}$ **(iii)** $\frac{1}{7-3i}$ **(iv)** $\frac{6+i}{i}$

6. Prove that the sum as well as product of complex numbers and its conjugate is a real number.

7. Write each expression as a complex number in the form $z = a + bi$.

(i) $(1-i)-2(4+i)^2$ **(ii)** $(1-i)^3$ **(iii)** $(2i)(8i)$ **(iv)** $(-6i)(-5i)^2$

8. Find the indicated absolute value of each complex number.

(i) $|3+4i|$ **(ii)** $|8-5i|$

9. If $z_1 = 3+2i$ and $z_2 = 4+5i$, then evaluate.

(i) $|z_1+z_2|$ **(ii)** $|z_1-z_2|$ **(iii)** $|z_1 z_2|$ **(iv)** $\left|\frac{z_1}{z_2}\right|$

10. Simplify and write your answer separately into real and imaginary parts.

(i) $\dfrac{2+3i}{5-2i}$ (ii) $\dfrac{(1+2i)^2}{1-3i}$ (iii) $\dfrac{1-i}{(1+i)^2}$

11. Show that $z.\bar{z}$ is a real number.

12. Show that $z=\bar{z}$ iff z is real.

13. Find the values of "a" & "b"

(i) $6a+i(4a-b)=12+3i$ (ii) $8a+i(5a-b)=16-7i$

(iii) $3a+i(a-b)=6+2i$ (iv) $4a+i(3a-b)=8-5i$

14. Find the values of "m" & "n"

(i) $m^2-2m+11ni=10-n^2i+14i$ (ii) $m^2-9m+8ni=n^2i+18i-7$

(iii) $m^2-4m+10ni=8-n^2i+6i$ (iv) $m^2-3m+7ni=n^2i+15i-5$

Student Learning Outcomes

- Solve the simultaneous linear equations with complex coefficients. For example,

$$5z-(3+i)w=7-i\,;\,(2-i)z+2iw=-1+i$$

- Write the polynomial $p(z)$ as a product of linear factors. For example,
  $z^2+a^2=(z+ia)(z-ia)$; $z^3-3z^2+z+5=(z+1)(z-2-i)(z-2+i)$
- Solve quadratic equation of the form $2pz+qz+r$ by completing squares, where p,q,r are real numbers and z a complex number. For example, solve:

$$z^2-2z+5=0\,;\,(z-1-2i)(z-1+2i)=0\,;\,z=1+2i,\ 1-2i$$

1.2 Solution of equations

To find the solution of different equations in complex variables either with real or complex co-efficient, we use some techniques which we used to find the solution of simultaneous linear equations.

1.2.1 Solution of Simultaneous Linear Equations with Complex Co-efficients

Consider the following equation

$$z+mw=n \longrightarrow (i)$$

where, m and n are complex numbers. The equation (i) is called a **linear equation** in two complex variables (or unknown) z and w.

Interesting Information

Solving equations with complex variables is essential in aerodynamics for designing aircraft wings. Engineers use complex potential flow theory to predict lift and drag forces, optimizing wing shape for performance and efficiency. This ensures safe and efficient air travel, advancing aviation technology and passenger safety.

$$z+m_1w=n_1$$
$$z+m_2w=n_2 \longrightarrow (ii)$$

These two equations together form a system of linear equations in two variables z and w.

The linear equations in two variables are also known as **simultaneous linear equations**.

For example

$$\left\{\begin{matrix} 5z-(3+i)w=7-i \\ (2-i)z+2iw=-1+i \end{matrix}\right\} \longrightarrow (iii)$$

is a system of linear equations with complex co–efficients.

The **solution** of a system in two variables z and w is an ordered pair (z,w) such that both the equations in the system are satisfied. For example, consider system (iii).

The ordered pair (z,w) where $z=1+i$ and $w=2i$ is a solution of (iii) because if we replace z by $1+i$ and w by $2i$, then both the equations are satisfied. The process of finding all solutions of the system of equations is known as **solving the system**.

Here, we shall find solution of a system of two equations with complex co-efficients in two variables z and w. The simple rule for solving such system of equations is the "Method of **Elimination and Substitution**".

(i) If necessary, multiply each equation by a constant so that the co-efficient of one variable in equation is the same.

(ii) Add or subtract the resulting equations to eliminate one variable, thus getting an equation in one variable.

(iii) Solve the equation in one variable obtained in step-2.

(iv) Put the known value of one variable in either of the original equation in step-1 and solve for the other variable.

(v) Writing together the corresponding values of the variables in the form of ordered pairs gives solution of the system.

Example 1.10 Solve the simultaneous linear equations with complex co-efficients.

$$5z-(3+i)w=7-i$$
$$(2-i)z+2iw=-1+i$$

Solution: Since,

$$5z-(3+i)w=7-i \longrightarrow (i)$$

$$(2-i)z+2iw=-1+i \longrightarrow \text{(ii)}$$

Multiplying equation (i) by $(2-i)$ we have

$$5(2-i)z-(3+i)(2-i)w=(7-i)(2-i)$$

$$\Rightarrow 5(2-i)z-(6-3i+2i-i^2)\ w=14-7i-2i+i2$$

$$\Rightarrow 5(2-i)z-(6-i+1)w=14-9i-1(i2=-1)$$

$$\Rightarrow 5(2-i)z-(7-i)w=13-9i \longrightarrow \text{(iii)}$$

Multiplying equation (ii) by 5, we have

$$5(2-i)z+10iw=-5+5i \longrightarrow \text{(iv)}$$

Subtracting equation (iii) from equation (iv), we have

$$5(2-i)z+10iw=-5+5i$$
$$\underline{-5(2-i)z\mp(7-i)w=-13\mp 9i}$$

$$10iw+(7-i)w=-18+14i$$

$$\Rightarrow (7+9i)w=-18+14i$$

$$\Rightarrow w=\frac{-18+14i}{7+9i}$$

$$\Rightarrow w=\frac{-18+14i}{7+9i}\times\frac{7-9i}{7-9i} \quad \text{(by rationalization)}$$

$$\Rightarrow \quad w=\frac{260i}{130}=2i$$

By putting the value of w in (i), we have

$$5z-(3+i)(2i)=7-i$$

$\Rightarrow \quad 5z-(6i+2i^2)=7-i$

$\Rightarrow \quad 5z-(6i-2)=7-i$

$\Rightarrow \quad 5z=7-i+6i-2$

$\Rightarrow \quad 5z=5+5i$

$\Rightarrow \quad z=\dfrac{5+5i}{5}=1+i$

Thus, $z=1+i$ and $w=2i$ is the solution of the simultaneous linear equations.

1.2.2 Expressing Polynomial $P(z)$ as a Product of Linear Factors

We are concerned with finding the linear factors of the following two types of polynomials.

(i) $P(z)=z^2+a^2$, where a is a real number.

(ii) $P(z)=az^3+bz^2+cz+d$ where a, b, c and d are real numbers respectively.

In factorizing polynomials of type (i) we simply use the fact that $i^2=-1$ in order to find linear factors.

For example,
$P(z)=z^2+a^2=z^2-i^2a^2=(z+ia)(z-ia)$.

However, in factorizing polynomials of type (ii), we use the factor theorem which has already been proved in our previous classes.

The factor theorem: Let $P(x)$ be any polynomial. Then $x-a$ is a factor of $P(x)$ if and only if $P(a)=0$

The method for factorizing the polynomials of type (ii) into linear factors is explained through the following example.

Example 1.11 Factorize the polynomial $P(z)=z^3+5z^2+32z-41$ into linear factors.

Solution: In factorizing the given polynomial $P(z)$ into linear factors, we use the factor theorem. To do so, we note that $z=1$ is a root of $P(z)$, since

$$P(1)=(1)^3+7(1)^2+33(1)-41$$
$$=1+7+33-4\ =0_1$$

By factor theorem $z-1$ is a factor of $P(z)$. We therefore arrange the terms in such a way that we can find a common factor $z-1$ as follows:

$P(z)=z^3+7z^2+33z-41$

$=(z^3-1)+(7z^2+33z-40)$

$=(z-1)(z^2+z+1)+(7z^2-7z+40z-40)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$=(z-1)(z^2+z+1)+(7z^2-7z)+(40z-40)$

$=(z-1)(z^2+z+1)+7z(z-1)+40(z-1)$

$=(z-1)[\ \ ][(z^2+z+1)+7z+40]$

$=(z-1)(z^2+8z+41)$

$=(z-1)(z^2+8z+16+25)$

$=(z-1)[(z^2+8z+16)+25]$

$=(z-1)[(z+4)^2-(5i)^2]$

$=(z-1)[(z+4)+5i][(z+4)-5i]$

$=(z-1)(z+4+5i)(z+4-5i)$

1.2.3 Solve quadratic equation of the form $pz^2+qz+r=0$ by completing square, where p, q and r are real numbers and z is a complex number

Consider the quadratic equation of the form

$pz^2+qz+r=0 \longrightarrow$ (i)

where p,q,r are real numbers; $p\neq 0$ and z is a complex variable.

We see that $z^2-2z+5=0$, $2z^2-8z+5=0$ and $z^2=0$ are all examples of quadratic equation in the variable z.

Solution of Quadratic Equations

All those values of z for which the given equation is true are known as the **solutions** or **roots** of the equation, and the set of all solutions is known as the **solution set**.

For example, $z^2+9=0$ or $z^2-(3i)^2=0$ is true only for $z=3i$ or $z=-3i$, hence $z=3i$ and $z=-3i$ are the solutions or roots of the given quadratic equation and $\{3i,-3i\}$ is the solution set.

To find the solutions of equations of the form (i), we use a method known as "Completing the Square" which is described as follows:

(i) Write the quadratic equation in its standard form.

(ii) Divide both sides of the equation by the co–efficient of z^2 if it is other than 1.

(iii) Shift the constant term to the right-hand side of the equation.

(iv) Add a number which is the square of half of the coefficient of z to both sides of the equation.

(iv) Write the left hand side of the equation as a perfect square and simplify the right hand side.

(v) Take square root of both sides of the equation and solve the resulting equation to find the solutions of the equation.

Example 1.12 Solve the quadratic equation.

$$9z^2+z+27=0$$

Solution: Since,

$$9z^2+z+27=0 \longrightarrow \text{(i)}$$

Divide equation (i) by 9.

$$z^2+\frac{1}{9}z+3=0$$

$$z^2+2\left(\frac{1}{18}\right)z=-3$$

$$z^2+2\left(\frac{1}{18}\right)z+\left(\frac{1}{18}\right)^2=-3+\left(\frac{1}{18}\right)^2$$

$$\left(z+\frac{1}{18}\right)^2=-3+\frac{1}{324}$$

$$\left(z+\frac{1}{18}\right)^2=-\frac{972}{324}+\frac{1}{324}$$

$$\left(z+\frac{1}{18}\right)^2=-\frac{971}{324}$$

Taking square root on both sides

$$z=\pm\frac{\sqrt{971}i}{18}-\frac{1}{18}$$

$$z=\frac{-1\pm\sqrt{971}i}{18}$$

Thus the solution , of the given equation is

$\dfrac{-1\pm\sqrt{971}i}{18}$ and the solution set is

$$\left\{\frac{-1+\sqrt{971}i}{18},\frac{-1-\sqrt{971}i}{18}\right\}.$$

Skill 1.2

Solving Simultaneous Linear Equations with Complex Coefficients:
Proficiency in solving sets of simultaneous linear equations involving complex coefficients .

Factoring Polynomials into Linear Factors:
Skill in expressing polynomials as products of linear factors, enabling the representation of polynomial equations in simpler forms.

Solving Quadratic Equations using Completing the Square Method:
Ability to solve quadratic equations of the form $pz^2+qz+r=0$ by completing squares, where p, q, r are real numbers and z is a complex number.

Exercise 1.2

1. Solve the simultaneous linear equations with complex co-efficients.

(i) $z+2w=4i$; $2z+5w=3$

(ii) $2z-w=3i+5$; $z+3w=12-4i$

(iii) $2z+(3+i)w=13i$; $(3-i)z-2w=2i$

(iv) $3z+w=3+2i$; $5z-w=11+3i$

2. Find solutions to the following equations.

(i) $z^2+z+3=0$ (ii) $z^2-1=z$

(iii) $z^2-2z+i=0$ (iv) $z^4+4=0$

3. Find solutions to the following equations.

(i) $z^3=1$ (ii) $z^3=8$

(iii) $(z-1)^3=-1$ (iv) $z^4+z^2+1=0$

4. Find the roots of the polynomial $p(z)=z^2-2z+2$ and use this to Factor the polynomial. Verify the factorization by expanding it.

5. Show that $1+i$, $1-i$ and 2 are roots of the polynomial $p(z)=z^3-4z^2+6z-4$ and use this to factor the polynomial $z^2-8(1-i)z+63-16i=0$.

6. Solve the given quadratic equation and write the solutions in the form $z=a+bi$.

(i) $z^2+2z+2=0$ (ii) $6z^2-5z+5=0$

(iii) $2z^2+z+3=0$ (iv) $3z^2+2z+4=0$

7. Show that each $z_1=-1+i$ and $z_2=-1-i$ does not satisfy the equation $z^2+2z+1=0$

8. Determine whether $1+2i$ is a solution of $z^2-2z+5=0$.

Student Learning Outcomes

- Explain the polar coordinates system
- Describe the polar representation of a complex number
- Apply the operations with complex numbers in polar representation
- Demonstrate simple equations and in-equations involving complex numbers in polar form.

1.3 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x,y), we get a unique point in the xy-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x+iy$ which corresponds to the ordered pair (x,y) can be represented geometrically as the unique point $P(x,y)$ in the xy-plane and vice-versa.

The plane having a complex number assigned to each of its point is called the complex plane or the Argand plane.

Obviously, in the Argand plane, the modulus of the complex number $|z|=r=\sqrt{x^2+y^2}$ is the distance between the point $P(x,y)$ and the origin O (0, 0) as shown in the figure 1.5.

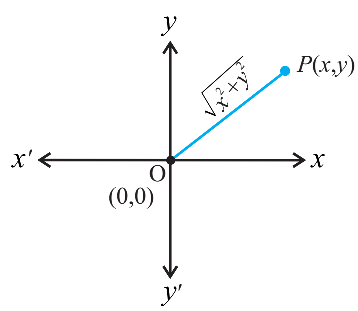


Figure 1.5

The points on the x-axis corresponds to the complex numbers of the form $a+0i$ and the points on the y-

axis corresponds to the complex numbers of the form $0 + \boldsymbol{ib}$.

The x-axis and y-axis in the Argand plane are called the real axis and the imaginary axis, respectively.

The representation of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ in the Argand plane are, the points $P(x, y)$ and $Q(x, -y)$, respectively.

Geometrically, the point $(x, -y)$ is the mirror image (Conjugate) of the point (x, y) on the real axis as shown in figure 1.6.

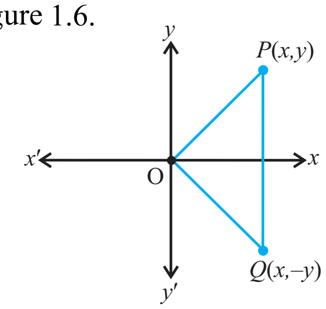


Figure 1.6

1.3.1 Polar representation of a complex number

Let point P represent the non-zero complex number $z = x + iy$. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis as shown in figure 1.7.

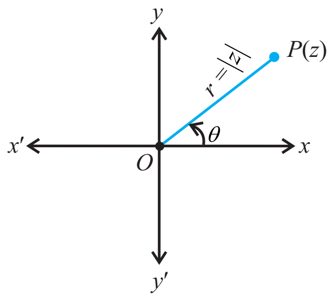


Figure 1.7

We may note that the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the polar coordinates of the point P. We consider the origin as the pole and the positive direction of the x axis as the initial line.

Let us understand the relation between rectangular and polar coordinates with a little derivation (figure 1.8).

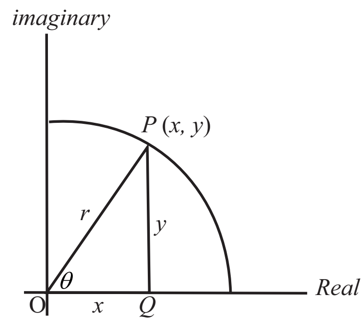


Figure 1.8

From the figure observe the right-angle triangle OQP. From there we get

$$cos(\theta) = \frac{x}{r}, \quad x = rcos(\theta)$$

$$sin(\theta) = \frac{y}{r}, \quad y = rsin(\theta)$$

$$tan(\theta) = \frac{y}{x}, \quad x^2 + y^2 = r^2$$

$$\theta = tan^{-1}\frac{y}{x}$$

With this in mind we can write $z = r(cos\theta + isin\theta)$. The latter is said to be the polar form of the complex number.

Here, $r = \sqrt{x^2 + y^2} = |z|$ is the modulus of z and θ is called the argument (or amplitude) of z which is denoted by arg z.

The argument of z is denoted by θ, which is measured in radians. However, there is an ambiguity in definition of the argument. The problem is that $sin(\theta + 2\pi) = sin\theta$, $cos(\theta + 2\pi) = cos\theta$, since the sine and the cosine are periodic functions of θ with period 2π. Thus θ is defined only up to an additive integer multiple of 2π. It is common practice to establish a convention in which θ is defined to lie within an interval of length 2π. The most common convention, which we adopt, is to take $-\pi < \theta \leq \pi$.

Principal arg (z): The argument θ which satisfies the inequality $-\pi<\theta\le\pi$ is known as the principal argument of z. This is denoted by Pr. arg (z) or Arg (z). In light of previous discussions it is tempting to identify arg z with arctan $\left(\frac{y}{x}\right)$. However, the real function arctan x is a multi-valued function for real values of x. It is conventional to introduce a single-valued real arctangent function, called the principal value of the arctangent, which is denoted by Arctan x and satisfies $-\frac{1}{2}\pi \le \arctan x \le \frac{1}{2}\pi$. Since $-\pi <$ arg $z \le \pi$, it follows that arg z cannot be identified with arctan $\left(\frac{y}{x}\right)$ in all regions of the complex plane. The correct relation between these two quantities is easily ascertained by considering the four quadrants of the complex plane separately illustrated in the figure 1.9 below:

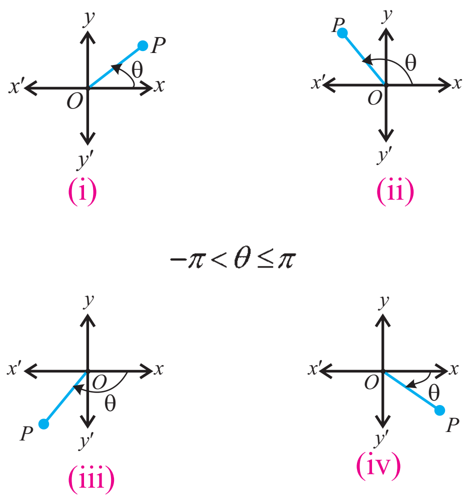


Figure 1.9

Rule to find Arg (z) (Principal value) :

Let $z=a+ib=(a,b)$ & $\tan^{-1}\left|\frac{b}{a}\right|=\theta$. Then

$Arg(z)=tan^{-1}\left(\frac{b}{a}\right)$ always gives the principal value.

It depends upon the quadrant in which the point (a, b) lies

a. $Arg(z)=\tan^{-1}\left|\frac{b}{a}\right|$ when z lies in 1st Quadrant

b. Arg(z)$=\pi-tan^{-1}\left|\frac{b}{a}\right|$, when z lies in 2nd Quadrant

c. Arg(z)$=tan^{-1}\left|\frac{b}{a}\right|-\pi$ lies in 3rd Quadrant

d. Arg(z)$=-tan^{-1}\left|\frac{b}{a}\right|$ $or\,2\pi-tan^{-1}\left|\frac{b}{a}\right|$ when z lies in 4th Quadrant

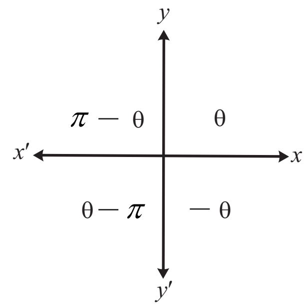


Figure 1.10

Principle value of z w.r.t quadrants

Properties of Arg (z):

- **i.** Arg (any +ve real no.) = 0
- **ii.** Arg (any –ve real no.) = π
- **iii.** $\text{Arg}\left(z-\bar{z}\right)=\pm\frac{\pi}{2}$
- **iv.** Arg ($z_1.z_2$) = Arg (z_1) + Arg (z_2)
- **v.** $\text{Arg}\left(\frac{z_1}{z_2}\right)=$ Arg (z_1) − Arg (z_2)

  $\text{Arg}\left(\bar{z}\right)=-\text{Arg}\,(z)=\text{Arg}\left(\frac{1}{z}\right)$
- **vi.** $\text{Arg}\,(z)+\text{Arg}\left(\bar{z}\right)=0$

Example 1.13 Represent the complex number $z=1+\sqrt{3}i$ in the polar form.

Solution

$x=1$; $y=\sqrt{3}$

$r=\sqrt{x^2+y^2}$

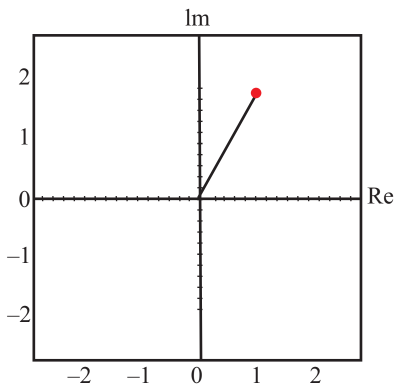
we get

$r=\sqrt{4}=2$ (conventionally, r >0) gives

$$\theta = \tan^{-1}\left[\frac{\sqrt{3}}{1}\right] = \theta = \frac{\pi}{3}$$

Therefore, required polar form is

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$



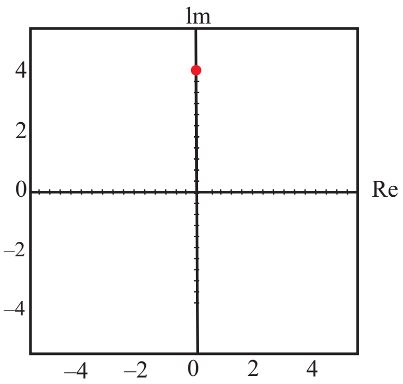
Example 1.14 Express the complex number $4i$ using polar coordinates.

Solution:

On the complex plane, the number $4i$ is a distance of 4 from the origin at an angle of $\frac{\pi}{2}$,

so $4i = 4\cos\frac{\pi}{2} + i4\sin\frac{\pi}{2}$

Note that the real part of this complex number is 0.



Example 1.15 Convert the complex number $\frac{-16}{1+i\sqrt{3}}$ polar form.

Solution:

The given complex number

$$\frac{-16}{1+i\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$= \frac{-16\left(1-i\sqrt{3}\right)}{1-(i\sqrt{3})^2} = \frac{-16\left(1-i\sqrt{3}\right)}{1+3}$$

$$= -4\left(1-i\sqrt{3}\right) = -4 + i4\sqrt{3}$$

Let $-4 = rcos\theta, 4\sqrt{3} = rsin\theta$

By squaring and adding, we get

$$16 + 48 = r^2\left(cos^2\theta + sin^2\theta\right)$$

which gives

$$r = 8$$

Hence $cos\theta = -\frac{1}{2}, sin\theta = \frac{\sqrt{3}}{2}$,

$$\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$$

$$\theta = \pi - \tan^{-1}\left|\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right|$$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$x + yi = r\cos\theta + i\,r\sin\theta$$

Thus, the required polar form is $8\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$

1.3.2 De Moivre's theorem

De Moivre's Theorem is a fundamental concept in mathematics that offers a method for raising complex numbers to a power using trigonometry.

The Derivation and Working of De Moivre’s Theorem can be understandable by a simple example

$$z^2 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2$$

$$z^2 = \cos^2\frac{\pi}{3} + i^2\sin^2\frac{\pi}{3} + 2i\cos\frac{\pi}{3}\sin\frac{\pi}{3}$$

$$z^2 = \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3} + i\left(2\sin\frac{\pi}{3}\cos\frac{\pi}{3}\right)$$

∵Trigonometric Identities
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$z^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \quad \text{in}$$

$$z = (r,\theta) = \left(1, \frac{\pi}{3}\right)$$

$$z^2 = (r^2, 2\theta) = \left(1, \frac{2\pi}{3}\right)$$

Remember that any complex number $z = x + yi$ can be written in the form of an ordered pair (r,θ) where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$ If

$$z = x + yi = r\cos\theta + i\, r\sin\theta$$

$$z^2 = (r\cos\theta + ir\sin\theta)^2$$

$$z^2 = r^2(\cos^2\theta - \sin^2\theta + i2\cos\theta\sin\theta)$$

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

Now, we can conclude that $z^n = r^n(\cos n\theta + i\sin n\theta)$ in (r,θ) can be represented as $(r^n, n\theta)$ and De Moivre's Theorem states that this is true for any rational number " n ".

De Moivre's

An important theorem in complex numbers is named after the French mathematician, **Abraham De Moivre (1667-1754).** Although born in France, he came to England where he made the acquaintance of Newton and Halley and became a private teacher of Mathematics. He never obtained the university position he sought but he did produce a considerable amount of research, including his work on complex numbers.

1.3.3 Euler's Theorem

The Euler form of complex numbers is a way to represent complex numbers using trigonometric functions. It's a useful and compact representation that involves the exponential function and imaginary unit. The Euler form is expressed as:

$$z = r\cdot e^{i\theta}$$

The relationship between the Cartesian form of a complex number $(a+bi)$ and the Euler form is given by:

$$z = a + bi = r\cdot(cos(\theta) + i\cdot sin(\theta)) = r\cdot e^{i\theta}$$

This form is closely related to De Moivre's theorem, which states that for any complex number $z = r\cdot e^{i\theta}$, raising it to a power n yields:

$$z^n = r^n\cdot e^{in\theta}$$

This form provides an elegant and efficient way to manipulate complex numbers and perform various operations, especially involving powers and roots.

Example 1.16 Write these polar coordinates in rectangular coordinates also express them in Euler's notation $r = 1,\ \theta = \frac{\pi}{6}$

Solution:

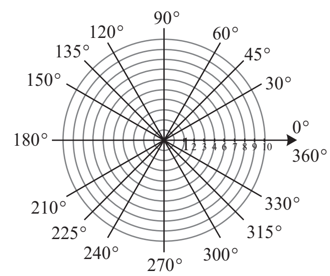
$$e^{i\theta} = cos\theta + i\ sin\theta$$

$$e^{i\frac{\pi}{6}} = cos\frac{\pi}{6} + i\ sin\frac{\pi}{6}$$

$$e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Interesting Information

Polar graphs



Representing data and functions that have a natural circular symmetry, such as the patterns of wind direction or the orbits of celestial bodies.

Example 1.17 Write the complex number 3 + 4i in Euler's form.

Solution:

To turn 3 + 4i into re^{ix} form,we do a Cartesian to Polar conversion:

$$r = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$x = \tan^{-1}\frac{4}{3} = 0.927\,(\text{to 3 decimals})$

So 3 + 4*i* can also be $5e^{0.927i}$

Example 1.18 Write $3e^{\frac{\pi}{6}i}$ in complex $a+bi$ form

Solution:

$$3e^{\frac{\pi}{6}i} = 3\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= 3\left[\frac{\sqrt{3}}{2} + i.\frac{1}{2}\right]$$

$$= \frac{3\sqrt{3}}{2} + i\frac{1}{2}$$

Example 1.19 Find the polar form of $-4+4i$

Solution:

On the complex plane, this complex number would correspond to the point $(-4,4)$ on a Cartesian plane.

We Can find the distance r and angle θ as we did in the last section

$$r^2 = x^2 + y^2 \quad r^2 = (-4)^2 + (4)^2 \quad r = \sqrt{32} = 4\sqrt{2}$$

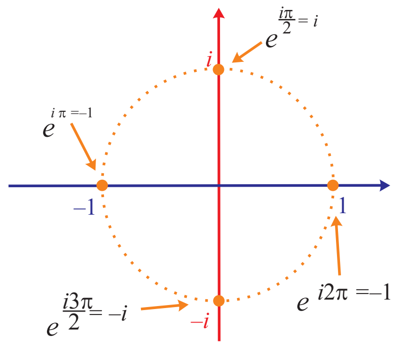
To find θ we can use $\theta = \pi - \tan^{-1}\left|\frac{4}{4}\right| = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

as θ lies in 2nd Quadrant

$$r = 4\sqrt{2},\ \theta = \frac{3\pi}{4} \qquad z = re^{i\theta} = 4\sqrt{2}e^{\frac{3\pi}{4}i}$$

Interesting Information

Euler's theorem is crucial in securing online transactions. It ensures that sensitive data, like credit card information, can be encrypted and decrypted securely between the buyer and the online store.

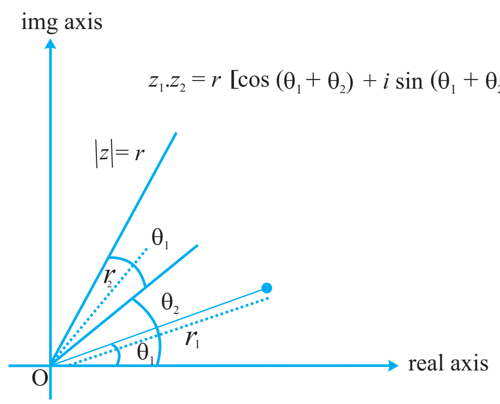


1.3.4 Products Of Complex Numbers In Polar Form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$z_1z_2 = r_1r_2\,cis(\theta_1+\theta_2)$$



Notice that the product calls for multiplying the moduli and adding the angles.

Example 1.20

Multiply $4(Cos\,30^o + iSin30^o)$ by $2(Cos\,60^o + isin\,60^o)$

Solution:

Let $z_1 = 4(Cos30^o + iSin30^o)$

and $z_2 = 2(Cos60^o + iSin60^o)$

Then

$$z_1.\,z_2 = 4\times 2\left[Cos(30^o+60^o) + iSin(30^o+60^o)\right]$$

$$z_1.\,z_2 = 8(Cos90^o + iSin90^o) = 8cis90^o$$

Note

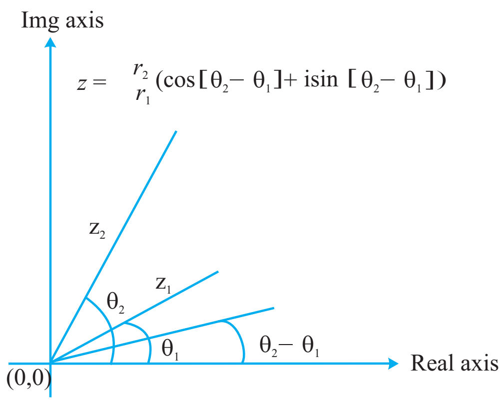
The term "cisθ" is a shorthand notation in mathematics representing the complex number expression cos(θ)+*i*sin(θ). This notation simplifies the handling of complex numbers, particularly in operations like multiplication, division, and exponentiation, by encapsulating trigonometric functions. Originating from the combination of the words cosine and sine, "cis" is heavily used in fields such as electrical engineering and physics to streamline calculations and reduce formula complexity. This notation is directly related to Euler's formula, which links exponential functions with trigonometric functions in the complex plane.

1.3.5 Division of Complex Numbers In Polar Form

If $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $z_2 = r_2(cos\theta_2 + isin\theta_2)$, then the quotient of these numbers is given as:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}cis(\theta_1 - \theta_2)$$



Example 1.21 Solve $\dfrac{8cis540^o}{2cis225^o}$

Solution:

$$\frac{8cis540^o}{2cis225^o} = 4cis(540^o - 225^o)$$

$$= 4\ cis(315^o)$$

$$= 4(Cos315^o + iSin315^o)$$

Since $Cos315^o = \dfrac{1}{\sqrt{2}}$ and $Sin315^o = -\dfrac{1}{\sqrt{2}}$

So $\dfrac{8cis540^o}{2cis225^o} = 4\left(\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}\right)$

Applications of Polar Coordinates:

Polar coordinates play a significant role in daily life across various fields. They are integral to navigation through GPS systems, architectural design for circular structures, and radar technology for tracking moving objects. Meteorologists use polar coordinates for weather analysis, while astronomers apply them to locate celestial bodies. Surveyors employ them for land mapping, artists create radial patterns, and mechanical engineers analyze rotating machinery. Even in culinary arts, polar coordinates help divide circular objects evenly. In diverse contexts, polar coordinates provide a valuable framework for describing and solving problems involving angles, distances, and radial patterns.

Example 1.22 Find the quotient of $1+\sqrt{3}i$ and $1+i$

Solution: We first write each of them in polar form.

Let

$$z_1 = 1+\sqrt{3}i \quad and \quad z_2 = 1+i$$

$r_1 = \sqrt{1+3} = 2 \quad and \quad r_2 = \sqrt{1+1} = \sqrt{2}$

$\theta_1 = tan^{-1}\sqrt{3} = 60^\circ \quad and \quad \theta_2 = tan^{-1}(1) = 45^\circ$

So, $z_1 = 2\left(Cos60^\circ + iSin60^\circ\right)$ and

$z_2 = \sqrt{2}\left(Cos45^\circ + iSin45^\circ\right)$

Now,

$$\frac{z_1}{z_2} = \frac{2\left(Cos60^\circ + iSin60^\circ\right)}{\sqrt{2}\left(Cos45^\circ + iSin45^\circ\right)}$$

$$= \sqrt{2}\left[Cos\left(60^\circ - 45^\circ\right) + iSin\left(60^\circ - 45^\circ\right)\right]$$

$$= \sqrt{2}\left[Cos15^\circ + iSin15^\circ\right] = \sqrt{2}cis15^\circ$$

Visual Representation of Multiplication of Complex Number in Polar Form

Skill 1.3

Understanding Polar Representation:
Interpret and demonstrate comprehension of the polar representation of complex numbers.
Applying Operations in Polar Form:
Apply operations $(+,-,\times,\div)$ with complex numbers proficiently in polar representation.

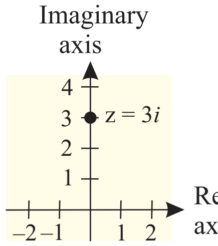
Exercise 1.3

1. Write the number in polar form with argument between $-\pi$ and π.

   (i) $-3+3i$ (ii) $1-\sqrt{3}I$

   (iii) $3+4i$ (iv) $8i$

2. In Exercises Write the complex number in trigonometric form

(i)


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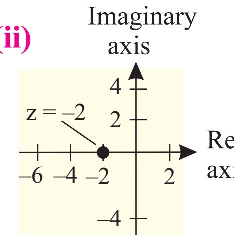
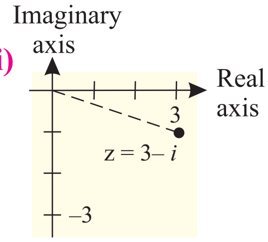
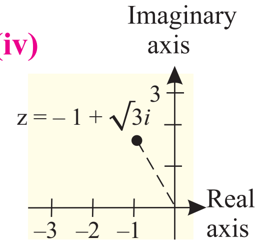


(iii)


(iv)


3. Find the indicated power using De Moivre's Theorem.

   (i) $\left(1+i\right)^{20}$ (ii) $\left(2\sqrt{3}+2i\right)^5$

   (iii) $\left(1-i\sqrt{3}\right)^5$ (iv) $\left(1-i\right)^8$

4. Write the number in the form of $a+bi$

   (i) $e^{\frac{\pi}{2}i}$ (ii) $e^{2\pi i}$ (iii) $e^{\frac{\pi}{3}i}$

   (iv) $e^{-\pi i}$ (v) $e^{2+i\pi}$ (vi) $e^{\pi+i}$

5. Express each of the following complex numbers in rectangular form

   (i) $3cis\dfrac{\pi}{4}$ (ii) $7cis\pi$ (iii) $8cis\dfrac{\pi}{2}$

   (iv) $10cis0.41$

6. Perform the indicated operations and give the results in polar form.

   (i) $\left(1+i\right)\left(1-\sqrt{3}i\right)$ (ii) $\left(-\sqrt{3}-i\right)\left(-1+i\right)$

   (iii) $(1+i)^4$ (iv) $(1-i\sqrt{3})^2$

   (v) $\sqrt{\dfrac{1+i}{1-i}}$ (vi) $\dfrac{-1-i}{-1+i}$

   (vii) $\dfrac{\left(1+i\sqrt{3}\right)\left(\sqrt{3}+1\right)}{1+i}$

7. **Perform the indicated operations in each and express the results in the form $a+ib$.**

   (i) $\left[4\left(Cos29^\circ + iSin29^\circ\right)\right]\left[\dfrac{1}{2}\left(Cos16^\circ + iSin16^\circ\right)\right]$

(ii) $\left(\sqrt{5}cis28^{\circ}\right)\left(\sqrt{5}cis8^{\circ}\right)\left(2cis9^{\circ}\right)$

(iii) $\dfrac{6\left(Cos51^{\circ}+iSin51^{\circ}\right)}{2\left(Cos21^{\circ}+iSin21^{\circ}\right)}$

(v) $\dfrac{10\left(Cos143^{\circ}+iSin143^{\circ}\right)}{5\left(Cos8^{\circ}+iSin8^{\circ}\right)}$

(vi) $\dfrac{\left(cis180^{\circ}\right)\left(6cis99^{\circ}\right)}{3\left(Cos39^{\circ}+iSin39^{\circ}\right)}$

(vii) $\dfrac{15\left(Cos48^{\circ}+iSin48^{\circ}\right)}{\left(3cis46^{\circ}\right)\left(2cis32^{\circ}\right)}$

1.4 Equations and in-equations involving polar form

In the world of math, complex numbers in polar form are like a special code that helps us solve equations differently. By understanding this code, we can tackle both simple equations and inequalities in a unique way. In this section, we'll dive into the world of complex numbers and polar form to unlock their secrets in solving mathematical puzzles.

Example 1.23 Solve the equation $z^2=1+i$ for z in polar form.

Solution:

First, express $1+i$ in polar form. Its magnitude is $\sqrt{1^2+1^2}=\sqrt{2}$, and its argument is $\dfrac{\pi}{4}$ (since it forms a 45-degree angle in the complex plane).

1. So, $1+i=\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$.

2. The equation becomes $z^2=\sqrt{2}e^{\frac{i\pi}{4}}$.

3. Taking the square root of both sides, we get

$$z=\pm\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{\frac{1}{2}}.$$

4. This simplifies to $z=\pm\sqrt[4]{2}e^{\frac{i\pi}{8}}$, which are the two solutions in polar form.

Example 1.24

Find all complex solutions to $z^3=8$.

Solution:

Since we are trying to solve $z^3 = 8$, we can solve for z as $z=8^{\frac{1}{3}}$. Certainly, one of these solutions is the basic cube root, giving $z = 2$. To find others, we can turn to the polar representation of 8.

Since 8 is a real number, which is in the complex plane on the horizontal axis at an angle of 0, giving the polar form $8e^{0i}$. Taking the $\dfrac{1}{3}$ power of this gives the real solution:

$$\left(8e^{0i}\right)^{\frac{1}{3}}=8^{\frac{1}{3}}\left(e^{0i}\right)^{\frac{1}{3}}=2e^{0}=2\cos(0)+i2\sin(0)=2$$

However, since the angle 2π is coterminal with the angle of 0, we could also represent the number 8 as $8e^{2\pi i}$. Taking the $\dfrac{1}{3}$ power of this gives a first complex solution:

$$\left(8e^{2\pi i}\right)^{\frac{1}{3}}=8^{\frac{1}{3}}\left(e^{2\pi i}\right)^{\frac{1}{3}}=2e^{\frac{2\pi}{3}i}=2\cos\left(\frac{2\pi}{3}\right)+i2\sin\left(\frac{2\pi}{3}\right)$$

$$=2\left(-\frac{1}{2}\right)+i2\left(\frac{\sqrt{3}}{2}\right)$$

$$=-1+\sqrt{3}i$$

For the third root, we use the angle of 4π, which is also coterminal with an angle of 0

$$\left(8e^{4\pi i}\right)^{\frac{1}{3}}=8^{\frac{1}{3}}\left(e^{4\pi i}\right)^{\frac{1}{3}}=2e^{\frac{4\pi}{3}i}=2\cos\left(\frac{4\pi}{3}\right)-i2\sin\left(\frac{4\pi}{3}\right)$$

$$=2\left(-\frac{1}{2}\right)-i2\left(\frac{\sqrt{3}}{2}\right)$$

$$=-1-\sqrt{3}i$$

Example 1.25 Solve $z^4=16i$

Solution:

To solve the equation $z^4=16i$ step by step, we'll first convert the right-hand side to polar form, and then find the fourth roots of the complex number.

1. **Convert $16i$ to Polar Form**:

- The magnitude (r) is the absolute value of $16i$, which is 16.
- The argument (θ) is the angle made with the positive real axis. Since $16i$ lies on the positive imaginary axis, θ is $\frac{\pi}{2}$ (or 90 degrees).
- Therefore, in polar form,
  $16i = 16\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ or $16e^{\frac{i\pi}{2}}$.

2. **Set Up the Equation in Polar Form**:

- The equation now is $z^4 = 16e^{\frac{i\pi}{2}}$.

3. **Find the Fourth Roots**:

- To find the fourth roots, we take the fourth root of the magnitude and divide the argument by 4.

  The fourth root of 16 is 2 (since $2^4 = 16$)

  Divide the argument $\frac{\pi}{2}$ by 4, $\frac{\pi}{2} \div 4 = \frac{\pi}{8}$.

  The general formula for the nth roots of a complex number in polar form is $re^{i(n\theta + 2k\pi)}$ where, $k = 0, 1, 2, \ldots, n-1$.

  For the fourth roots, k takes values 0, 1, 2, and 3.

4. **Calculate the Roots**:

For k=0: $z = 2e^{\frac{i\pi}{8}}$.

For k=1: $z = 2e^{i\left(\frac{\pi}{8}+\frac{\pi}{2}\right)} = 2e^{\frac{i5\pi}{8}}$.

For k=2: $z = 2e^{i\left(\frac{\pi}{8}+\pi\right)} = 2e^{\frac{i9\pi}{8}}$.

For k=3: $z = 2e^{i\left(\frac{\pi}{8}+\frac{3\pi}{2}\right)} = 2e^{\frac{i13\pi}{8}}$.

These are the four fourth roots of $16i$ in polar form. Each root represents a complex number that, when raised to the fourth power, equals to $16i$.

Polar Inequations

Polar inequalities, or polar inequations, are mathematical expressions used in the polar coordinate system to describe regions where an inequality is true. These inequalities are particularly useful in areas such as physics, engineering, and mathematics, especially when dealing with circular or spiral shapes

Polar Inequalities

A polar inequality looks like $r > f(\theta)$, $r < f(\theta)$, $r \geq f(\theta)$, or $r \leq f(\theta)$, where:

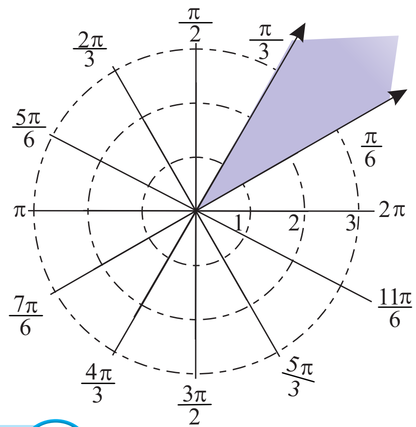
- r is the radial distance from the origin.
- θ is the angle.
- $f(\theta)$ is a function of θ.

These inequalities describe regions in the plane. For example, $r < 2$ describes all points inside a circle of radius 2, centered at the origin.

Example 1.26

Shade the region described by the polar inequalities $0 \leq r < \infty$, $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ In the shaded region below, θ is restricted to be between $\frac{\pi}{6}$ and $\frac{\pi}{3}$ (including those angles). Within that range for θ, all positive r values are allowed. Thus, the shaded region shown below is represented by the polar inequalities

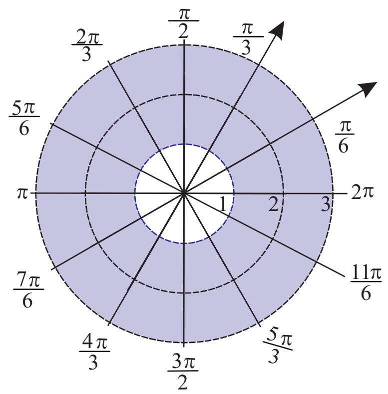
$0 \leq r < \infty$, $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$



Example 1.27 Shade the region described by the polar inequalities $1 < r < 2.5$, $0 \leq \theta \leq 2\pi$

Solution:

In the shaded region below, r is restricted to be between 11 and 2.5. Notice that we do not want to include the points corresponding to r = 1 and r = 2.5, so we use dashed lines instead of solid ones. Any value of θ between 0 and 2π is allowed. Thus, the washer shaped region shaded below is represented by the polar inequalities $1 < r < 2.5$, $0 \le \theta \le 2\pi$



Example 1.28

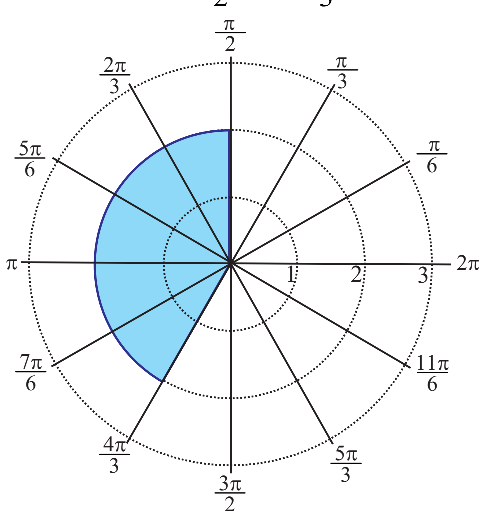
Write a set of polar inequalities to describe the shaded region shown below.

The shaded region begins at the origin and has an outer radius of 2. Therefore, $0 \le r \le 2$

The shaded region stretches from the angle $\frac{\pi}{2}$ to the angle $\frac{4\pi}{3}$, so $\frac{\pi}{2} \le \theta \le \frac{4\pi}{3}$.

Putting together these bounds for r and θ gives us the polar inequalities,

$$0 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{4\pi}{3}$$



Example 1.29 Convert the following equations and inequalities in Cartesian form:

$$0 \le \arg\left(\frac{z-3}{2+i}\right) \le \frac{\pi}{4}.$$

To solve this question, which is $0 \le \arg\left(\frac{z-3}{2+i}\right) \le \frac{\pi}{4}$,

we need to understand what this inequality is asking is to convert it from polar to cartesian coordinates. This inequality specifies that the angle of the complex number formed by the expression $\frac{z-3}{2+i}$ is between 0 and $\frac{\pi}{4}$ radians. Let's break this down step-by-step:

Step 1: Simplify the Expression

First, let's write z as $x + yi$, where x and y are the real and imaginary parts of z, respectively. Then, substitute this into the given complex fraction:

$$z - 3 = (x + yi) - 3 = (x - 3) + yi$$

Now, divide by $2 + i$. To simplify, multiply the numerator and the denominator by the conjugate of the denominator:

.

$$\frac{(x-3)+yi}{2+i}\cdot\frac{2-i}{2-i} = \frac{(x-3)(2-i)+yi(2-i)}{(2+i)(2-i)}$$

$$= \frac{(x-3)2-(x-3)i+2yi-yi^2}{4+1}$$

$$= \frac{2x-6-xi+3i+2y-y}{5}$$

$$= \frac{(2x-6+2y)+(-x+3+2y)i}{5}$$

Let's denote the real part as u and the imaginary part as v.

$$u = \frac{2x-6+2y}{5}$$

$$v = \frac{-x+3+2y}{5}$$

Step 2: Interpret the Inequality

The inequality $0 \le \arg\left(\frac{z-3}{2+i}\right) \le \frac{\pi}{4}$ means the angle formed by $u+vi$ lies in the first quadrant between the positive x-axis and the line $y=x$

Step 3: Convert to Cartesian Inequality

The angle condition translates into inequalities involving u and v :

1. $v \ge 0$ since the angle must be non-negative.
2. $\tan(\arg(u+vi)) \le 1$, which simplifies to $\frac{v}{u} \le 1$.

Substitute the expressions for u and v :

$$0 \le \tan^{-1}\left|\frac{\frac{-x+3+2y}{5}}{\frac{2x-6+2y}{5}}\right| \le \frac{\pi}{4}$$

$$\tan^{-1}\left|\frac{\frac{-x+3+2y}{5}}{\frac{2x-6+2y}{5}}\right| \le \frac{\pi}{4}$$

Taking "tan" on both sides we have

$$\frac{-x+3+2y}{2x-6+2y} \le 1$$

$$-x+3+2y \le 2x-6+2y$$

$$3x \ge 9$$

$$x \ge 3$$

$$0 \le \tan^{-1}\left|\frac{\frac{-x+3+2y}{5}}{\frac{2x-6+2y}{5}}\right|$$

Taking "tan" on both sides we have

$$-x+3+2y \ge 0$$

Now assuming the minimum value from $x \ge 3$ which is "x=3"

$$-3+3+2y \ge 0$$

$$y \ge 0$$

Conclusion

For the complex number $z = x + yi$ to satisfy the given polar inequality, the real part x must be at least 3 , and the expression and value of y must be at least 0.This defines a specific region in the complex plane where the original inequality holds

Applications of Polar Inequalities

1. Navigation and GPS:

Polar inequalities define safe travel zones and avoid obstacles.

Example: A ship must stay within 10 km of a lighthouse but avoid 30° to 60° from the north:

$$r \le 10$$

$$\theta \notin \left[30^\circ, 60^\circ\right]$$

2. Radar and Sonar Systems:

Define detection range and sector for radar and sonar.

Example: Detect objects within 15 km and between 45° and 135° :

$$r \le 15$$

$$45^\circ \le \theta \le 135^\circ$$

3. Astronomy:

Focus telescopes on specific sky regions, avoiding light pollution.

Example: Observe within 20° of a star, excluding 100° to 120° :

$$r \le 20^\circ$$

$$\theta \notin \left[100^\circ, 120^\circ\right]$$

4. Geographical Zoning:

Urban planners define zones for construction and traffic management.

Example: Buildings within 5 km of the center, restricted to 0° to 90° :

$$r \le 5$$

$$0^\circ \le \theta \le 90^\circ$$

5. Military Defense:

Define safe zones and target areas in military operations.

Example: Monitor within 50 km, excluding 200° to 240° :

$$r \le 50$$

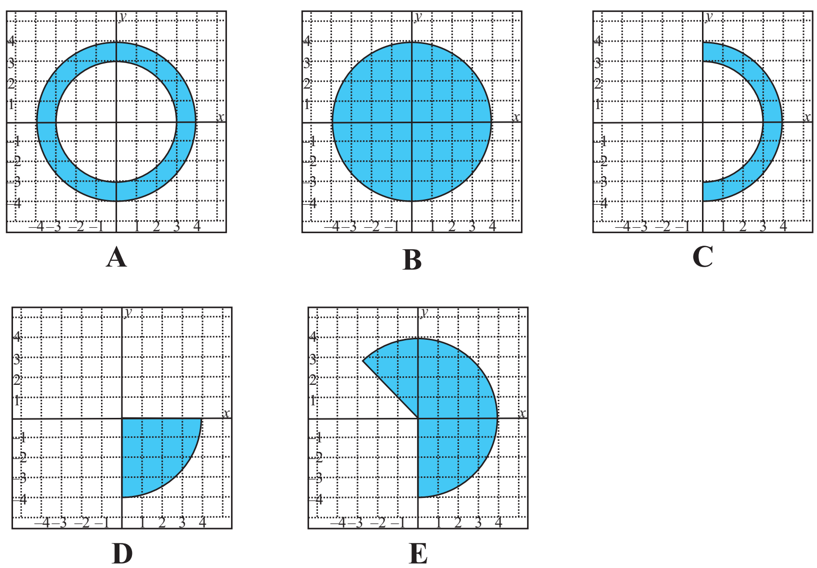
$$\theta \notin \left[200^\circ, 240^\circ\right]$$

Skill 1.4

Application to Equations:
Demonstration of equations & inequations involving complex numbers in polar form.

Exercise 1.4

1. Solve $z=\sqrt{1+i}$ in polar form.
2. Find the polar form of the complex number $z=-1-i$.
3. Solve $z^2=-4$ in polar form.
4. If $z=3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$, find z^3.
5. Determine the modulus and argument of z if $z^2=2-2i$.
6. Solve for z in the equation $z^3=8i$.
7. If $z_1=2e^{\frac{i\pi}{4}}$ and $z_2=4e^{\frac{i\pi}{3}}$, find $\frac{z_1}{z_2}$ in polar form.
8. Solve $z^4=-16$ in polar form.
9. Solve the equation $z^5=32\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ for all values of z in polar form.
10. Write the Polar inequalities describes by the following Polar Region



11. Shade the region described by the polar inequalities $2<r<4$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.
12. Convert the following inequalities in to possible simplest Cartesian form:

(i) $-\frac{\pi}{2}\le\arg(z+3i)\le 0$ (ii) $z\bar{z}\le 16e^{i\frac{\pi}{2}}$ (iii) $-\frac{\pi}{4}\le\arg(z+2-i)\le\frac{\pi}{4}$

Student Learning Outcomes

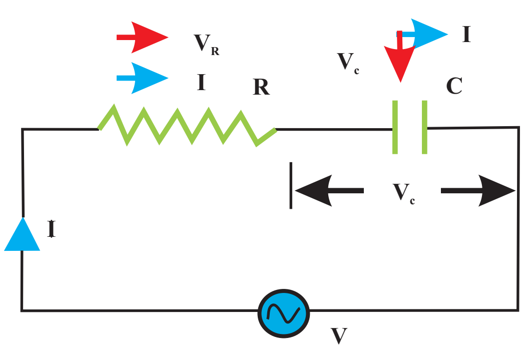
✧ Apply concepts of complex numbers to real world problems (such as cryptography, wave phenomena, calculate voltage, current, circuits, the velocity and pressure of the fluid).

1.5 Applications of Complex Numbers

Step into the real world of complex numbers, where we'll see how they help solve practical problems in areas like electronics, waves, and fluid dynamics. This section connects classroom theory with real-life applications, showing how complex numbers are key in understanding and solving everyday challenges in various fields

Example 1.30 Voltage Across a Capacitor in an RC Circuit

In an AC circuit with a series combination of a 100-ohm resistor and a 50 μF capacitor, find the voltage across the capacitor. The circuit is connected to a 200V, 60 Hz AC supply.



Hints:

Inductive Reactance: $Xc = \dfrac{1}{\omega C}$ where ω=2πf

Total Impedance in Polar Form: $Z = R - jX_c$,

with magnitude $|Z| = \sqrt{R^2 + X_c^2}$ and phase angle

$\theta = arctan\left(\dfrac{X_c}{R}\right)$

Current in AC Circuit: $I = \dfrac{V}{Z}$

Voltage across Inductor: $V_c = I \times jX_c$

Solution:

1. **Calculate the Capacitive Reactance (X_C):**

$$X_C = \frac{1}{2\pi f C}$$

$$f = 60\ Hz,\ C = 50 \times 10^{-6}\ F$$

$$X_C = \frac{1}{2\pi \times 60 \times 50 \times 10^{-6}} \text{ ohms}$$

$$X_C \approx 53.05\ ohms$$

2. **Calculate the Total Impedance (Z) in Polar Form**:

$$Z = R - jX_C$$

$$Z = 100 - j53.05\ ohms$$

Magnitude:

$$|Z| = \sqrt{100^2 + (-53.05)^2} \approx 114.9\ ohms$$

Phase Angle: $\theta = arctan\left(-\dfrac{53.05}{100}\right) \approx -27.8^o$

3. **Calculate the Current (I)**:

$$I = \frac{V}{Z}$$

$$I = \frac{200\angle 0^o}{114.9\angle -27.8^o}$$

$$I \approx 1.74\angle 27.8^o\ A$$

4. **Calculate the Voltage Across the Capacitor (V_C):**

$$V_C = I \times jX_C$$

$$V_C = 1.74\angle 27.8^o \times 53.05\angle 90^o$$

$$V_C \approx 92.3\angle 117.8^o\ volt$$

Conclusion:

The voltage across the capacitor is approximately $92.3\angle 117.8^o$ volts.

Example 1.31

If a fluid's velocity at a point is represented by the complex number $5e^{-\frac{i\pi}{4}}$ m/s, determine its horizontal and vertical velocity components.

Solution:

The complex number for the velocity is in polar form; $5e^{-\frac{i\pi}{4}}$. In polar form a complex number is represented as $re^{i\theta}$, where r is the magnitude and θ is the angle in radians.

Convert to Rectangular Form:

The rectangular form of a complex number is $a+bi$, where a is the real part and b is the imaginary part. To convert from polar to rectangular form, use the formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Calculate the Real Part (Horizontal Component):

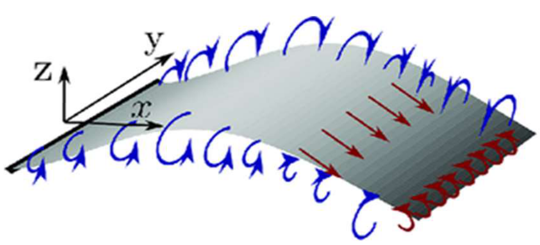
$$x = 5\cos\left(-\frac{\pi}{4}\right)$$

$$x \approx 5 \times 0.707 \Rightarrow x \approx 3.54 \text{ m/s}$$

Calculate the Imaginary Part (Vertical Component):

$$y = 5\sin(-\pi/4)$$

$$y \approx -5 \times 0.707 \Rightarrow y \approx -3.54 \text{m/s}$$



Example 1.32

In a wave simulation, an electromagnetic wave's phase and amplitude are represented by $2-i$. Find the inverse of this representation and interpret its physical meaning

Solution:

Given complex representation: $2 - i$

Find the complex conjugate: $2 + i$ (inverse representation)

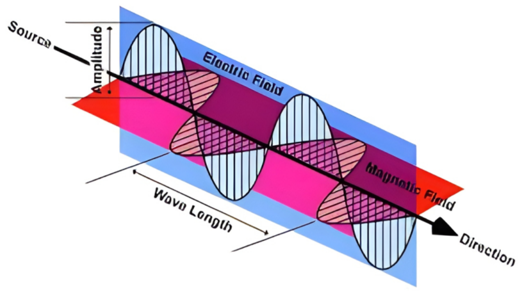
Calculate amplitude:

For both representations: $\sqrt{2^2 + (-1)^2} = \sqrt{5}$

Calculate phase:

Original representation: $arctan\left(\frac{1}{-2}\right) \approx -0.4636$ radians or approximately -26.57^o degrees

Inverse representation: $arctan\left(\frac{1}{2}\right) \approx 0.4636$ radians or approximately 26.57 degrees



Interpretation:

The amplitudes of both representations are the same (5). The phase of the inverse representation is the negative of the phase of the original representation, suggesting a 180-degree (π radians) phase shift in the electromagnetic wave.

Skill 1.5

Applying Concepts in Real-World Contexts:

Apply complex number concepts practically to real-world scenarios such as cryptography, wave phenomena, electrical circuits, fluid dynamics etc.

Exercise 1.5

1) A wave is represented by the complex number $5-3i$. Calculate the magnitude of this wave.
2) Convert the impedance $4+4i$ ohms of an electrical component into polar form.
3) If two signals are represented by $2+3i$ and $3-i$, find the product of these signals.
4) Divide the complex voltages $10+6i$ V by $2-4i$ V and express your answer in rectangular form.
5) If a mechanical wave is represented by the complex number $4e^{\frac{i\pi}{2}}$, describe its amplitude and phase shift.
6) If a fluid's velocity at a point is represented by the complex number $27e^{\frac{i5\pi}{6}}$ m/s, determine its horizontal and vertical velocity components.
7) Given an AC circuit with an impedance of $8+6i$ ohms and a voltage of $10e^{\frac{i\pi}{3}}$ volts. Calculate the current using Ohm's Law
8) A fluid's velocity at a point is represented by the complex number $5-2i$ m/s. What is the angle of the flow direction with respect to the horizontal
9) In an AC circuit, the voltage V and current I can be represented as complex numbers. If the voltage across a circuit is represented by $5+12i$ volts and the current by $2+3i$ amperes, calculate the power of the circuit.
10) Imagine you have a simple encryption system where each letter of the alphabet is assigned a unique complex number. If 'A' is assigned $1+2i$, 'B' is $2+3i$, and so on, what would be the complex number representation of the word "AB"
11) In an AC circuit with a series combination of a 75-ohm resistor and a 0.15 H inductor, find the voltage across the inductor. The circuit is connected to a 150V, 50 Hz AC supply.

Hints:

Inductive Reactance: $X_L = \omega L$ where ω=2πf

Total Impedance in Polar Form: $Z = R + jX_L$,

with magnitude $|Z| = \sqrt{R^2 + X_L^2}$ and phase angle

$\theta = arctan\left(\frac{X_L}{R}\right)$

Current in AC Circuit: $I = \frac{V}{Z}$

Voltage across Inductor: $V_L = I \times jXL$

Review Exercise 1

1. Each of the questions or incomplete statement below is followed by four suggested answers or completions. In each case, select the one that is the best of the choices.

(i) If $x^2 = -9$ then $x =$ ________

(a) 3 (b) –3 (c) 3i (d) ±3i

(ii) The real part of complex number z = 7i is:

(a) 0 (b) 7 (c) –7 (d) 1

(iii) The imaginary part of complex number z = 8 + 10i:

(a) 0 (b) 10 (c) 20 (d) 8

(iv) The additive inverse of $3+\frac{1}{2}i$ is ________

(a) $\frac{2}{6+i}$ (b) $\frac{2}{6-i}$ (c) $-3-\frac{1}{2}i$ (d) $3-\frac{1}{2}i$

(v) The multiplicative identity of complex number is:

(a) 0 (b) 1 (c) 2 (d) 3

(vi) The additive identity of a complex number is:

(a) 0 (b) 1 (c) 2 (d) 3

(vii) $\dfrac{\sqrt{-1250}}{\sqrt{2}} =$ ______

(a) –25i (b) 25 (c) –25 (d) $\pm 25i$

(viii) $i^{10} =$ ______

(a) 1 (b) –1 (c) i (d) –i

(ix) The conjugate of 7 + 4i = ______

(a) –7 + 4i (b) 7 – 4i (c) –7–4i (d) 7 + 4i

(x) If we replace i by –i in z = x + iy then another complex number obtained is known as:

(a) primer factor of z (b) reciprocal of z

(c) additive inverse of z (d) complex conjugate of z

(xi) If $z_1 = 3 + i$ and $z_2 = 1 + 4i$ then $\text{Re}(z_1 - z_2) =$ ______

(a) –3 (b) 2 (c) 3 (d) 2

(xii) $|z_1 + z_2| \leq$ ______

(a) $|\bar{z}_1| + |z_2|$ (b) $|z_1| + |\bar{z}_2|$ (c) $|\bar{z}_1| + |\bar{z}_2|$ (d) $|z_1| + |z_2|$

(xiii) $x^2 + y^2 =$ ______

(a) $(x + yi)\,(x - yi)$ (b) $(x + y)\,(x - y)$ (c) $(x + yi)\,(x - y)$ (d) $(x + y)\,(x - yi)$

(xiv) If $|z^2| + 1 = |z^2 - 1|$ then z lies on:

(a) a circle (b) real axis (c) imaginary axis (d) None of the above

(xv) The conjugate of the complex number six x – i cos 2x is:

(a) sin x + i cos2x (b) cos x – i sin2x (c) –sin x – i cos2x (d) –sin x + i cos2x

(xvi) If z = – 1 – i, then arg z is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{5}{\pi}$ (c) $\dfrac{5\pi}{4}$ (d) 4π

(xvii) $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 + \left(\dfrac{1-i}{\sqrt{2}}\right)^8$

(a) 1 (b) 4 (c) 2 (d) 8

(xviii) If n is positive integer then $\left(\dfrac{1+i}{1-i}\right)^{4n+1} =$ ______

(a) 1 (b) –1 (c) i (d) $-i$

(xix) If $i^2=-1$, then $i+i^2+i^3+\ldots\ldots\ldots\ldots$ to 1000 terms is equal to

(a) 1 (b) -1 (c) i (d) 0

(xx) If $\dfrac{5(-8+6i)}{(1+i)^2}=a+ib$ then (a, b) equals

(a) $(15,20)$ (b) $(20,15)$ (c) $(-15,20)$ (d) $(-15,-20)$

2. Solve the following and write the result in standard form.

i) $-8i(2i-7)$ ii) $(-4-8i)(3+i)$ iii) $(7-5i)(-2-3i)$ iv) $\dfrac{2i}{1+i}$

v) $(8-4i)(-3+9i)$ vi) $\dfrac{2+3i}{2+i}$ vii) $\dfrac{3-4i}{4+3i}$ viii) $-\dfrac{6i}{3+2i}$

3. Evaluate x^2-2x+2 for $x=1+i$.

4. Evaluate x^2-7x-5 for $x=1-2i$.

5. Find E, the voltage of a circuit, if $I=(4-5i)$ amperes and $R=(3+7i)$ ohms.

6. Find E, the voltage of a circuit, if $I=(2-3i)$ amperes and $R=(3+5i)$ ohms.

7. The mathematician Girolamo Cardano is credited with the first use (in 1545) of negative square roots in solving the now-famous problem, “Find two numbers whose sum is 10 and whose product is 40.

8. Show that the complex numbers $5+i\sqrt{15}$ and $5-i\sqrt{15}$ satisfy the conditions of the problem.

9. Represent the Following Complex numbers on a Graph in standard Form

i) $5\left(\cos\dfrac{\pi}{9}+i\sin\dfrac{\pi}{9}\right)$ ii) $10\left(\cos\dfrac{2\pi}{9}+i\sin\dfrac{2\pi}{5}\right)$

iii) $3(\cos 165.5^o+i\sin 165.5^o)$ iv) $9(\cos 58^o+i\sin 58^o)$

10. Perform the Operation on the Following & Represent result in Trigonometric Form

i) $\left[2\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)\right]\left[6\left(\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}\right)\right]$

ii) $\left[0.45(\cos 310^o+i\sin 310^o)\right]\times\left[0.6(\cos 200^o+i\sin 200^o)\right]$

iii) $\dfrac{\cos 50^o+i\sin 50^o}{\cos 20^o+isin20^o}$

iv) $\dfrac{12(\cos 52^o+i\sin 52^o)}{3(\cos 110^o+i\sin 110^o)}$

11. Use DeMoivre’s Theorem to find the indicated power of the complex number

i) $(3-2i)^8$ ii) $2\left(\sqrt{3}+i\right)^7$ iii) $\left[2\left(\cos\dfrac{\pi}{10}+i\sin\dfrac{\pi}{10}\right)\right]^5$ iv) $\left[3(\cos 150^o+i\sin 150^o)\right]^4$

Summary

- **Representation of Complex Numbers:** Complex numbers are represented as $z=a+ib$ or (a,b), where a and b are real numbers, and $i=-1$.
- **Understanding Real and Imaginary Parts:** The real part of a complex number is a, and the imaginary part is b.
- **Condition for Equality:** Complex numbers are equal if and only if their real and imaginary parts are equal.
- **Complex Conjugate:** The complex conjugate of $z = a+ib$ is $z = a-ib$.
- **Absolute Value or Modulus:** The modulus of a complex number $z=a+ib$ is $|z|=\sqrt{a^2+b^2}$
- **Solving Simultaneous Linear Equations:** Complex numbers are used to solve simultaneous linear equations with complex coefficients.
- **Factoring Polynomials:** Polynomials can be factored into linear factors, e.g., $z^2+a^2=(z+ia)(z-ia)$ or $z^3-3z^2+z+5=(z+1)(z-2-i)(z-2+i)$.
- **Solving Quadratic Equations:** Quadratic equations of the form $pz^2+qz+r=0$ are solved using methods like completing the square, with p, q, r as real numbers and z as a complex number.
- **Understanding Polar Coordinates:** Polar coordinates are a coordinate system that represents points based on their distance and angle from a reference point (the pole).
- **Polar Representation of Complex Numbers:** Complex numbers can be represented in polar form as $z=r(\cos\theta+i\sin\theta)$, where r is the modulus, and θ is the argument (angle).
- **Polar Multiplication:** In polar form, complex numbers can be multiplied by multiplying their moduli and adding their arguments.
- **Polar Division:** In polar form, complex numbers can be divided by dividing their moduli and subtracting their arguments.
- **De Moivre's Theorem:** De Moivre's Theorem states that $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$, which is a powerful tool for raising complex numbers to integer powers.
- **Euler Form:** Euler's formula, $e^{i\theta}=\cos\theta+i\sin\theta$, connects complex exponentials to trigonometric functions and is essential in complex analysis.
- **Application to Equations:** Complex numbers in polar form and Euler's formula are used to solve equations involving complex quantities.
- **Real-World Application:** Complex number concepts find practical applications in various fields, including cryptography, wave phenomena, electrical circuits, fluid dynamics, and measurements of physical quantities like voltage, current, velocity, pressure, and more.

CHAPTER 2

Matrices and Determinants

Matrices are fundamental in the world of cryptography, where they are instrumental in ensuring the security of digital communication and data. In cryptography, matrices are indispensable for encoding and decoding messages, preserving the confidentiality and integrity of sensitive information. Cryptographers employ matrices to perform intricate mathematical transformations that obscure the original message, rendering it indecipherable to unauthorized individuals. These operations leverage matrix properties, such as multiplication and inversion. Matrices play a crucial role not only in data encryption but also in digital signatures, authentication and secure communication protocols. This chapter explores how matrices are the cornerstone of modern cryptography, safeguarding the digital realm and enabling secure online transactions, confidential messages and protected data.

Students' Learning Outcome

1. Apply matrix operations (addition/subtraction and multiplication of matrices) with real and complex entries.
2. Evaluate determinants of 3×3 matrix by using cofactors and properties of determinants.
3. Use row operations to find the inverse and the rank of a matrix.
4. Explain a consistent and inconsistent system of linear equations and demonstrate through examples
5. Solve a system of 3 by 3 nonhomogeneous linear equations by using matrix inversion method and Cramer's Rule.
6. Solve a system of three homogeneous linear equations in three unknowns using the
7. Gaussian elimination method.
8. Apply concepts of matrices to real world problems such as (graphic design, data encryption, seismic analysis, cryptography, transformation of geometric shapes, social network analysis).

Knowledge

1. **Matrix Operations:**
Scalar Multiplication, Addition, Subtraction of Matrices: Perform arithmetic operations (addition, subtraction, scalar multiplication) on matrices.
2. **Matrix Multiplication:** Understand and apply rules for multiplying matrices, both with real and complex entries.
3. **Determinant of a Square Matrix:** Define determinant, minor, and cofactor of an element of a matrix.
4. **Adjoint of a Matrix:** Know and use the adjoint method to calculate the inverse of a square matrix.
5. **Properties of Determinants:** State and prove properties of determinants for various operations.

Matrix Row Operations:

6. **Row and Column Operations:** Understand and perform row and column operations on matrices.
7. **Echelon and Reduced Echelon Forms:** Define and reduce a matrix to its echelon and reduced echelon forms.
8. **Rank of a Matrix**: Recognize the rank of a matrix using row operations.

Systems of Linear Equations:

9. **Homogeneous and Non-Homogeneous Equations:** Distinguish between homogeneous and non-homogeneous systems of linear equations in two and three unknowns.
10. **Consistent and Inconsistent Systems:** Define and demonstrate consistent and inconsistent systems of linear equations through examples.
11. **Solving Systems of Equations:** Solve systems of equations using various methods:
12. **Matrix Inversion Method**
13. **Gauss Elimination Method (Echelon Form)**
14. **Gauss-Jordan Method (Reduced Echelon Form)**
15. **Cramer's Rule for 3x3 systems.**

Application:

16. **Real-world Problem Solving:** Apply matrix operations and solution methods to real-world problems involving systems of equations and matrix manipulations.

Pre & Post Requisite

Class 10
Chapter # 2
Matrices & Determinants

Class 11
Chapter # 2
Matrices & Determinants

Skills

1. **Matrix Operations Proficiency:**
   - Conduct arithmetic operations on matrices accurately.
   - Apply rules for matrix multiplication confidently.
   - Understand the properties of matrices under addition and multiplication.
2. **Matrix Properties and Manipulation:**
   - Verify the transpose of matrix products.
   - Calculate determinants, minors, and cofactors for square matrices.
   - Recognize singular and non-singular matrices' significance.
3. **Understanding Properties of Determinants:**
   - Apply and prove properties of determinants for various matrix operations.
   - Evaluate determinants without expansion using determinant properties.
4. **Matrix Row Operations Mastery:**
   - Perform row and column operations on matrices proficiently.
   - Reduce matrices to echelon and reduced echelon forms effectively.
5. **Systems of Linear Equations Problem-Solving Skills:**
   - Differentiate between homogeneous and non-homogeneous equation systems.
   - Identify consistent and inconsistent equation systems.
   - Utilize multiple methodologies to solve equation systems efficiently.
6. **Real-world Application Proficiency:**
   - Apply matrix operations and solution methodologies to real-world problems involving linear transformations and modeling.

Introduction

Building on the basic concepts you've learned in earlier classes, we're ready to explore deeper into matrices, a pivotal element in higher mathematics that bridges simple equations and complex systems. In this chapter, we will explore advanced topics essential in both academic settings and practical applications across Mathematics, Science, and Engineering.

Matrices enable us to solve linear equations efficiently, perform geometric transformations, and even facilitate data encryption. We'll investigate the use of determinants and methods like Cramer's Rule, which provide precise solutions to intricate problems. This exploration will not only enhance your mathematical skills but also prepare you for further academic pursuits and professional challenges.

For example, the information regarding one day cricket matches played in a season between Pakistan and England is presented in the following table.

	Play	Win	Draw	Lost
Pakistan	13	4	3	6
England	13	6	3	4

The information is clear when presented in this way. If we want to know how many matches England lost against Pakistan, we go along the row 'England' and column 'Lost' and find that it is 4. As long as we remember what each number represents, we could remove the row and column headings and write just the numbers, enclosing them in square brackets or parenthesis such as

$$\begin{bmatrix} 13 & 4 & 3 & 6 \\ 13 & 6 & 3 & 4 \end{bmatrix}$$

2.1 Matrices

Matric and its Notations

A matrix is a rectangular array of numeric symbols or expressions arranged in rows and columns. e.g.

$$X = \begin{bmatrix} 8 & 4 & 1 & 3 \\ 8 & 3 & 1 & 4 \end{bmatrix}$$

is a matrix. Usually a matrix is represented by the capital letters and their entries by small letters.

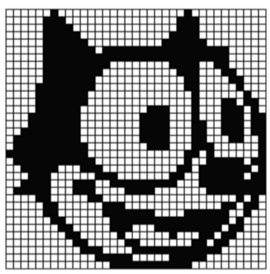
In general, an $m \times n$ matrix has the following rectangular array:

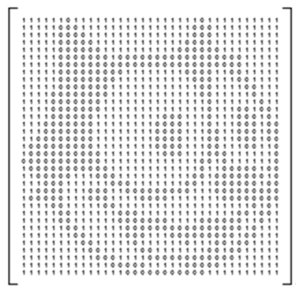
$$X = \left[x_{ij}\right]_{m\times n} = \begin{bmatrix} x_{11} & x_{12} & x_{13}\cdots & x_{1n} \\ x_{21} & x_{22} & x_{23}\cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & x_{m3}\cdots & x_{mn} \end{bmatrix}_{m\times n} 1 \le i \le m, 1 \le j \le n i, \quad i, j \in N$$

The element, x_{ij} is an element lying in the ith row and jth column and is known as the $(i, j)th$ element of X. The number of elements in an $m \times n$ matrix will be equal to mn.

Interesting Information

Matrices are essential for representing and manipulating data in various fields, such as computer graphics, economics, and genetics. They enable complex calculations and data analysis, making them invaluable tools in science and engineering.





35×35

Let us consider an example. Assume that we have to write the following results of different subjects in a 3 by 4 matrix where each row corresponds to a different student and each column corresponds to a different subject

Student A:	Math (85),	Science (92),
	English (78),	History (88)
Student B:	Math (76),	Science (89),
	English (95),	History (82)
Student C:	Math (90),	Science (84),
	English(88),	History(91)

Solution:

$$\begin{array}{c|cccc} & \text{Math.} & \text{Sci.} & \text{Eng.} & \text{Hist.} \\ \hline \text{Student A} & & & & \\ \text{Student B} & & \begin{bmatrix} 85 & 92 & 78 & 88 \\ 76 & 89 & 95 & 82 \\ 90 & 84 & 88 & 91 \end{bmatrix} & & \\ \text{Student C} & & & & \end{array}$$

$$\begin{array}{l c} & \begin{array}{cccc} \text{Math.} & \text{Sci.} & \text{Eng.} & \text{Hist.} \end{array} \\ \begin{array}{l} \text{Student A} \\ \text{Student B} \\ \text{Student C} \end{array} & \begin{bmatrix} 85 & 92 & 78 & 88 \\ 76 & 89 & 95 & 82 \\ 90 & 84 & 88 & 91 \end{bmatrix} \end{array}$$

Student Learning Outcomes

✧ Apply matrix operations (addition/subtraction and multiplication of matrices) with real and complex entries

2.2 Algebra of Matrices

In this section we set up an algebra of matrices, defining various operations of addition, subtraction, multiplication and so on, we have been familiar about these operations from out early grades.

Note

$A + A = 2A$, $A + A + A = 3A$ and in general, if n is a positive integer, then $\underbrace{A + A + \ldots + A}_{n\text{-times}} = nA$

2.2.1(a)Scalar Multiplication

If $A = [a_{ij}]$ is $m \times n$ matrix and k is a scalar, then the product of k and A, denoted by kA, is the matrix formed by multiplying each entry of A by k, that is, $kA = [ka_{ij}]$ whose $(i, j)th$ is ka_{ij} . Obviously, order of kA is $m \times n$.

If $A = [a_{ij}] \in M_{m\times n}$ (the set of all $m \times n$ matrices over the real field $\Re$ then $ka_{ij} \in \Re$, for all i and j, which shows that $kA \in M_{m\times n}$ .It follows that the set $M_{m\times n}$ possesses the closure property with respect to scalar multiplication. If $A, B \in M$ and r,s are scalars, then we can say that

$r(sA) = (rs)A$, $(r + s)A = rA + sA$, $r(A + B) = rA + rB$

For example, if $A = \begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix}$ and k is any scalar, then

$$kA = k\begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 2k & 4k \\ 8k & 3k \end{bmatrix}$$

2.2.1(b)Addition of Matrices

If A and B are $m \times n$ matrices, the sum of A and B, denoted by $A + B$, is the $m \times n$ matrix obtained by adding the corresponding entries of A and B; that is, $A + B$ is the $m \times n$ matrix whose $(i, j)th$ entry is $a_{ij} + b_{ij}$.

Example 2.1 Add A and B.

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}$$

Solution

$$A + B = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3 & 2+4 & 3+6 \\ 1+1 & -1+2 & 2+5 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 9 \\ 2 & 1 & 7 \end{bmatrix}$$

2.2.1(c)Subtraction of Matrices

If A and B are $m \times n$ matrices, the difference of A and B, denoted by $A - B$, is the $m \times n$ matrix obtained by subtracting the corresponding entries of A and B; that is, $A - B$ is the $m \times n$ matrix whose $(i, j)th$ entry is $a_{ij} - b_{ij}$.

Notice that the matrices A and B must have the same order for their sum and difference to be defined.

Example 2.2 If $A = \begin{bmatrix} 5 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix}_{2\times 3}$ and

$$B = \begin{bmatrix} 8 & 7 & 3 \\ 1 & 1 & 2 \end{bmatrix}_{2\times 3} \text{ then,}$$

Solution: $A - B = \begin{bmatrix} 5 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix}_{2\times 3} - \begin{bmatrix} 8 & 7 & 3 \\ 1 & 1 & 2 \end{bmatrix}_{2\times 3}$

$$= \begin{bmatrix} 5-8 & 2-7 & 4-3 \\ 0-1 & -1-1 & 3-2 \end{bmatrix}_{2\times 3} = \begin{bmatrix} -3 & -5 & 1 \\ -1 & -2 & 1 \end{bmatrix}_{2\times 3}$$

Example 2.3 Compute the matrices $A + B$, $3A - A$, and $3A + 4B$, where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix}$$

Solution: We have

$$A+B=\begin{bmatrix}-1 & 5 & 2\\ 7 & -9 & 1\end{bmatrix},$$

$$3A=\begin{bmatrix}9 & 12 & 6\\ 6 & -9 & 0\end{bmatrix}, -A=\begin{bmatrix}-3 & -4 & -2\\ -2 & +3 & 0\end{bmatrix}$$

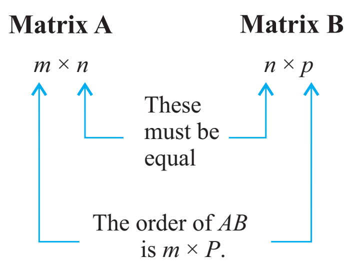
$$3A-A=\begin{bmatrix}6 & 8 & 4\\ 4 & -6 & 0\end{bmatrix} \text{ and}$$

$$3A+4B=\begin{bmatrix}9 & 12 & 6\\ 6 & -9 & 0\end{bmatrix}+\begin{bmatrix}-16 & 4 & 0\\ 20 & -24 & 4\end{bmatrix}$$

$$=\begin{bmatrix}-7 & 16 & 6\\ 26 & -33 & 4\end{bmatrix}$$

2.2.1(d)Multiplication of Matrices

Two matrices A and B are said to be comfortable for multiplication giving the product AB, if the number of columns of A is equal to the number of rows of B. In the product, A is called the pre-multiplier or pre-factor of B and B is called the post-multiplier or post-factor of A.



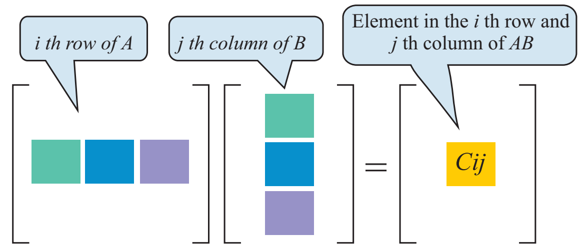
The entry in AB is obtained by multiplying the entries in i^{th} row of A by the corresponding entries in j^{th} column of B and adding the results.

For an $m \times n$ matrix $A=[a_{ij}]$ and an $n \times p$ matrix $B=[b_{ij}]$, the product $AB=[c_{ij}]$ is an $m \times p$ matrix, where $c_{ij}=a_{i1}\,.\,b_{1j}+a_{i2}\,.\,b_{2j}+a_{i3}\,.\,b_{3j}+\ldots+a_{in}\,.\,b_{nj}$

In other words, the entry c_{ij} in AB is obtained by multiplying the entries in row i of A by the corresponding entries in column j of B and adding the results.

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}=\sum_{k=1}^{n}a_{ik}b_{kj}$$

Look at the following matrices that illustrates the expression for c_{ij}



jth column

$$\begin{array}{c c} & \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1p}\\ a_{21} & a_{22}\cdots & a_{2p}\\ \vdots & \vdots & \vdots\\ a_{i1} & a_{i2}\cdots & a_{ip}\\ \vdots & \vdots & \vdots\\ a_{m1} & a_{m2}\cdots & a_{mp}\end{bmatrix}_{m\times p} \\ \end{array} \begin{bmatrix} b_{11} & b_{12}\cdots & b_{1j} & \cdots & b_{1n}\\ b_{21} & b_{22}\cdots & b_{2j} & \cdots & b_{2n}\\ \vdots & \vdots & \vdots & & \vdots\\ b_{p1} & b_{p2}\cdots & b_{pj} & \cdots & b_{pn}\end{bmatrix}_{p\times n}$$

ith row →

$$a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots\ldots+a_{ip}b_{pj}=c_{ij}$$

$$=\begin{bmatrix} c_{11} & c_{12}\cdots & c_{1j}\cdots & c_{1n}\\ c_{21} & c_{22}\cdots & c_{2j}\cdots & c_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ c_{i1} & c_{i2}\cdots & c_{ij}\cdots & c_{in}\\ \vdots & \vdots & \vdots & \vdots\\ c_{m1} & c_{m2}\cdots & c_{mj}\cdots & c_{mn}\end{bmatrix}_{m\times n}$$

Example 2.4 It is given that $C=[2\quad 3\quad 4]$ and

$$D=\begin{bmatrix}1\\ -1\\ 2\end{bmatrix}$$

Compute the products CD and DC

Solution: The orders of matrices C and D are 1×3 and 3×1 respectively; therefore, the product CD is possible, and results in a matrix of order 1×1 as shown below.

$$CD=[2\quad 3\quad 4]_{1\times 3}\begin{bmatrix}1\\ -1\\ 2\end{bmatrix}_{3\times 1}$$

$$=[(2)(1)+(3)(-1)+(4)(2)]=[7]$$

The orders for D and C are 3×1 and 1×3 respectively and therefore, the product DC is also possible. However, multiplication of these results in a 3×3 matrix as shown below.

$$DC = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}_{3\times1} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}_{3\times1}$$

$$= \begin{bmatrix} (1).(2) & (1).(3) & (1).(4) \\ (-1).(2) & (-1).(3) & (-1).(4) \\ (2).(2) & (2).(3) & (2).(4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & -4 \\ 4 & 6 & 8 \end{bmatrix}_{3\times3}$$

Note

(i) If A is a square matrix, then the product AA is defined and is denoted by A^2. We define $A^3 = A.A^2$ or $A^2.A$ and so on and in general $A^n = A^{n-1}.A$ or $A.A^{n-1}$.

(ii) We say that in AB, the matrix B is **pre-multiplied** (or multiplied from the left) by A, and is **post-multiplied** (or multiplied from the right) by B.

Example 2.5 Compute AB if $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix}$

and $B = \begin{bmatrix} 2 & i & 3 \\ 4 & 1 & -i \\ 1 & 2 & 3 \end{bmatrix}$

Solution:

$$AB = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & i & 3 \\ 4 & 1 & -i \\ 1 & 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2+12+2 & i+3+4 & 3-3i+6 \\ 8+4+2 & 4i+1+4 & 12-i+6 \\ 12+4+0 & 6i+1+0 & 18-i+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 16 & i+7 & 9-3i \\ 14 & 4i+5 & 18-i \\ 16 & 6i+1 & 18-i \end{bmatrix}$$

Example 2.6 **Softball Team Expenses**

Two Cricket teams submit equipment lists to their sponsors.

	Team A	Team B
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs 800 Rs., each ball costs 100 Rs., and each pair of gloves costs 1000 Rs.. Use matrices to find the total cost of equipment for each team.

Solution:

The equipment lists and the costs per item can be written in matrix form

$$A = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}, \qquad B = \begin{bmatrix} 800 & 100 & 1000 \end{bmatrix}$$

The total cost of equipment for each team is given by the product BA.

$$BA = \begin{bmatrix} 800 & 100 & 1000 \end{bmatrix} \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}$$

$$BA = \begin{bmatrix} (800\times12)+(100\times45)+(1000\times15) & (800\times15)+(100\times38)+(1000\times17) \end{bmatrix}$$

$$BA = \begin{bmatrix} 29100 & 32800 \end{bmatrix}$$

So, the total cost of equipment for the team A is 29100 Rs. and the total cost of equipment for the Team B is 32800 Rs.

Notice that you cannot find the total cost using the product AB because it is not defined. That is, the number of columns of (2 columns) does not equal the number of rows of (1 row)

2.2.2 Properties of Matrix Addition, Scalar Multiplication and Matrix Multiplication

(Properties of Matrix Addition and Scalar Multiplication)

Let A, B, and C be $m \times n$ matrices and let **s** and **i** be any scalars. Then

(i) $A+B=B+A$. (commutative law of matrix addition)

(ii) $(A+B)+C=A+(B+C)$ (associative law of matrix addition)

(iii) $A+O=A$. (Additive Identity O is a null matrix)

(iv) $A+(-A)=O$. (Additive Inverse $-A$ is negative of matrix A)

(v) $s(A+B)=sA+sB$. (Distributive Property multiplication over addition)

(vi) $(s+t)A=sA+tA$. (Distributive Property addition over multiplication)

Properties of Matrix Multiplication

For matrices A, B, and C, assuming that the indicated operations are possible:

$A(BC)=(AB)C$. Associative Property of Multiplication

$A(B+C)=AB+AC$ Distributive Property

$(B+C)A=BA+CA$. Distributive Property

Skill 2.1

Matrix Operations Proficiency:

- Conduct arithmetic operations on matrices accurately.
- Apply rules for matrix multiplication confidently.
- Understand the properties of matrices under addition and multiplication.

Exercise 2.1

1. Write the following products of matrices as a single matrix.

(i) $\begin{bmatrix} 1 & 0 & 7 \\ 3 & 2 & -1 \\ -5 & -2 & 5 \end{bmatrix}\begin{bmatrix} 4 & -6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 4 & -3 \\ -2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\left(\begin{bmatrix} 5 & 3 \\ -4 & 1 \end{bmatrix}+\begin{bmatrix} -1 & -5 \\ 6 & 8 \end{bmatrix}\right)$

(iv) $\begin{bmatrix} 5 & -6 & 7 \\ 8 & 2 & 3 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} 7 & 2 & 4 \\ 3 & 1 & 0 \\ 9 & -2 & 1 \end{bmatrix}$

2. If possible, find matrix A.

(i) $A\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 6 & 7 \end{bmatrix}$ **(ii)** $A\begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) $A\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}=\begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$ **(iv)** $A\begin{bmatrix} 5 & i \\ -i & 6 \end{bmatrix}=\begin{bmatrix} i & 5 \\ 6 & -i \end{bmatrix}$

3. Solve each of the following matrix equation for x and y.

(i) $\begin{bmatrix} x-6y \\ -x+2y \end{bmatrix}=\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

(ii) $\begin{bmatrix} -5 & x & 2 \\ 4y & 0 & 3 \end{bmatrix}=\begin{bmatrix} x & -5 & 8 \\ 0 & y & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} -4x+2 & 3 & 7 \\ 5 & 3y+4 & -1 \end{bmatrix}=\begin{bmatrix} 6x & 3 & 7 \\ 5 & 2y+4 & -1 \end{bmatrix}$

4. Write the following sums as a single matrix.

(i) $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}+\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} -5 & 2 & 4 \end{bmatrix}+\begin{bmatrix} 1 & 0 & -8 \end{bmatrix}$

(iii) $\begin{bmatrix} -2 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & -2 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 7 \\ 2 & 1 & 3 \end{bmatrix}+\begin{bmatrix} 1 & 3 & 1 \\ 8 & 6 & 5 \\ 2 & 1 & 7 \end{bmatrix}$

5. Write the following product as a single matrix.

(i) $-2\begin{bmatrix} -4 & 2 & 3 \\ -3 & 6 & -5 \end{bmatrix}$ **(ii)** $\begin{bmatrix} -2 & -7 & -3 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \\ 8 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & -8 & 7 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 6 & -1 \\ 0 & 7 \end{bmatrix}$ **(iv)** $\dfrac{1}{2}\begin{bmatrix} 4 & 3 & -9 \\ 14 & -5 & 13 \\ 0 & 9 & 6 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & -3 & 1 \\ 0 & 5 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 7 & 0 \\ -1 & -2 & 5 \\ 6 & 2 & -1 \end{bmatrix}$

6. Compute the indicated matrices, where

$A=\begin{bmatrix} 2 & -1 & 5 \\ 3 & 4 & 1 \end{bmatrix},\ B=\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 4 \end{bmatrix}$.

(i) $4A$	**(ii)** $-A$	**(iii)** $4A-2B$
(iv) $3A+2B$	**(v)** $(2B)^t$	**(vi)** A^t+2B^t
(vii) $A+B$	**(viii)** $(A+2B)^t$	**(ix)** A^t
(x) $A-B$	**(xi)** $-(B^t)$	**(xii)** $(-B)^t$

7. Prove that $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies $A^2-4A-5I=0$.

8. Let $A=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ B=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$,

then verify the following:

(i) $A^2=B^2=C^2=-I$ (ii) $AB=-BA=-C$

9. $A=\begin{bmatrix} 6 & 0 & 7 \\ -2 & 6 & 8 \\ 3 & 1 & 3 \end{bmatrix}, B=\begin{bmatrix} 6 & 4 & 4 \\ 4 & 3 & 0 \\ 0 & 3 & 3 \end{bmatrix}, C=\begin{bmatrix} 3 & 2 & 6 \\ 4 & 5 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ and

a,b are real numbers, then verify the following:

(i) $A+B=B+A$	**(ii)** $A+(B+C)=(A+B)+C$
(iii) $A+O=O+A=A$	**(iv)** $A+(-A)=(-A)+A=$ O
(v) $(ab)A=a(bA)$	**(vi)** $a(A+B)=aA+aB$
(vii) $(a+b)A=aA+bA$	**(viii)** $A(BC)=(AB)C$
(ix) $A(B+C)=AB+AC$	**(x)** $(A+B)C=AC+BC$

10. Determine whether commutative property w.r.t. multiplication holds in each of the following cases or not.

(i) $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix},\ B=\begin{bmatrix} a & b \\ -c & d \end{bmatrix}$

(ii) $A=\begin{bmatrix} 0 & 5 & 2 \\ 2 & 1 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 4 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$

11. Let $A=\begin{bmatrix} 4 & 3 & 0 \\ 3 & 0 & 9 \end{bmatrix}$ Show that

(i) $(A^t)^t=A$ **(ii)** $AA^t\neq A^tA$

12. Solve the following matrix equations for X.

(i) $X-3A=2B,\ if\ A=\begin{bmatrix} 3 & 4 & 3 \\ -2 & 2 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 7 & 3 & 4 \\ 3 & -1 & 4 \end{bmatrix}$

(ii) $2(X-A)=B,\ if\ A=\begin{bmatrix} 1 & 5 & 8 \\ 3 & -1 & 2 \end{bmatrix},\ B=\begin{bmatrix} 4 & 6 & 2 \\ 0 & -4 & 2 \end{bmatrix}$

13. If A$=\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix},\ B=\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ then find X such that $A+2X=B$.

14. Find x if $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & x \\ -2 & -3 & 3 \end{bmatrix}=$ O

15. A manufacturer of electronics produces three models of portable CD players, which are shipped to two warehouses. The number of units of model i that are shipped to warehouse j is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 5000 & 4000 \\ 6000 & 10000 \\ 8000 & 5000 \end{bmatrix}$$ the Prices per unit are represented in Rs by the Matric $B = \begin{bmatrix} 45 & 60 & 35 \end{bmatrix}$. Computer BA and interpret the result.

Student Learning Outcomes

✧ Evaluate determinants of 3 × 3 matrix by using cofactors and properties of determinants

2.3 Determinants

It is a scalar value that can be calculated from the element of a square matrix using certain properties of matrix. Determinate of a given matrix X is usually dented by det $[X]$ or $|X|$

Consider a square matrix X of order n given by

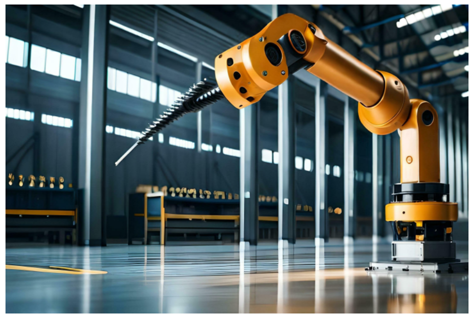
$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix} \longrightarrow \text{(i)}$$

The associated determinant of X is denoted by

$$|X| = \begin{vmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} \longrightarrow \text{(ii)}$$

Some determinants of higher order can be evaluated only after much tedious calculations. The more calculation is involved, the greater the chance of error. In this section we will describe a procedure for evaluating the determinants of order $n \geq 3$. We first find minors and cofactors of matrices in order to evaluate determinants.

Interesting Information



In robotics, determinants are used to calculate the orientation and position of robotic arms. Determinants help solve systems of equations that define the movement and rotation of the arm, ensuring precise control and accurate positioning in tasks like assembly lines and medical surgeries.

2.3.1 Determinants of square matrix

A. In case of 2 × 2 matrix the determinant can be defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

B. In case of 3 × 3 matrix the determinant can be defined as.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{11}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}(a_{22}\, a_{33} - a_{23}a_{32}) - a_{22}(a_{21}\, a_{33} - a_{23}\, a_{31}) + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31})$

$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{33} - a_{22}\, a_{21}\, a_{33} + a_{22}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31}$

Example 2.7 If $A = \begin{bmatrix} -3 & 6 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 7 \\ 7 & 3 \end{bmatrix}$, then

det(A)=$|A|$=

$\begin{vmatrix} -3 & 6 \\ -1 & 5 \end{vmatrix} = (-3)(5) - (-1)(6) = -15 + 6 = -9$ and det $(B) = |B| =$

$$\begin{vmatrix} 3 & 7 \\ 7 & 3 \end{vmatrix} = (3)(3) - (7)(7) = 9 - 49 = -40$$

2.3.2 Minor and Co-factor of an Element of a Matrix or its Determinants.

A. Minor of an Element

For a square matrix $X = [x_{ij}]$, the minor M_{ij} of an element x_{ij} is the determinant of the matrix formed by deleting the i^{th} row and the j^{th} column of X.

$$\text{If } X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \text{then}$$

$$\text{minor of } x_{11} = M_{11} = \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} \text{obtained as} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

$$M_{11} = x_{22}\,x_{33} - x_{32}\,x_{23}$$

$$\text{minor of } x_{23} = M_{23} = \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \text{obtained as} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

$$M_{23} = x_{11}\,x_{32} - x_{31}\,x_{12}$$

Example 2.8 Let $A = \begin{bmatrix} 8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$,

Find each of the following.

(i) M_{11} (ii) M_{23}

Solution:

(i) For M_{11}, we delete the first row and the first column then find the determinant of the 2 × 2 matrix formed by the remaining elements.

$$\begin{bmatrix} 8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix} \qquad M_{11} = \begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix}$$

$= (-6)5 - (-3)7$

$= -30 + 21$

$= -9$

For M_{23}, we delete the second row and the third column and find the determinant of the 2×2 matrix formed by the remaining elements.

$$\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix} \qquad M_{23} = \begin{vmatrix} -8 & 0 \\ -1 & -3 \end{vmatrix}$$

$= -8(-3) - (-1)0$

$= 24$

B. Cofactor of an Element

For a square matrix $X = [x_{ij}]$, the cofactor a_{ij} of an element x_{ij} is given by $X_{ij} = (-1)^{i+j} M_{ij}$, Where M_{ij} is the minor of x_{ij}.

$$\text{Thus, if } X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \text{then}$$

$$\text{Cofactor of } x_{11} = X_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix}$$

$= 1 \times (x_{22}\,x_{33} - x_{23}\,x_{32})$

$= x_{22}\,x_{33} - x_{23}\,x_{32}$

Sign pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

***n* × *n* matrix**

Example 2.9 For the matrix

$$X = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix},$$

(i) X_{11} (ii) X_{23}

Solution:

(i) $X_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-9) = (1)(-9) = -9$.

(ii) $X_{23} = (-1)^{2+3} M_{23} = (-1)^5(24) = (-1)(24) = -24$.

2.3.3 Determinant of a Square Matrix

Let X be a square matrix of order $n(\geq 3)$ given by

$$|A| = \begin{vmatrix} x_{11} & x_{12} & x_{1j} & x_{1n} \\ x_{21} & x_{22} & x_{2j} & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i1} & x_{i2} & x_{ij} & x_{in} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{nj} & x_{nn} \end{vmatrix} \longrightarrow \text{(i)}$$

The determinant $|X|$ of the matrix X is defined to be the sum of the products of each element of row (or column) and its co-factor, that is

$$|X| = x_{i1}X_{i1} + x_{i2}X_{i2} + \ldots\ldots + x_{in}X_{in}; i = 1,2\ldots\ldots,n \rightarrow \text{(ii)}$$

Or

$$|X| = x_{1j}X_{1j} + x_{2j}X_{2j} + \ldots\ldots + x_{nj}X_{nj}; j = 1,2\ldots\ldots,n \rightarrow \text{(iii)}$$

If we put $i = 1$ in (ii), we get

$|X| = x_{11}X_{11} + x_{12}X_{12} + \ldots\ldots + x_{1n}X_{1n}$ which is called the expansion of $|X|$ by first row (or w.r.t. first row).

Similarly, if we put j=1 in (iii), we get

$|X| = x_{11}X_{11} + x_{21}X_{21} + \ldots\ldots + x_{n1}X_{n1}$ which is called the expansion of $|X|$ by first column and so on. Thus, if X is a square matrix of order 3, that is

$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$, then by (ii) and (iii), we have

$$|X| = x_{i1}X_{i1} + x_{i2}X_{i2} + x_{i3}X_{i3}; \quad i = 1,2,3 \longrightarrow \text{(iv)}$$

or $|X| = x_{1j}X_{1j} + x_{2j}X_{2j} + x_{3j}X_{3j}; \quad j = 1,2,3 \longrightarrow \text{(v)}$

For example, if $i = 2$, then by (iv), we have

$|X| = x_{21}X_{21} + x_{22}X_{22} + x_{23}X_{23}$. This can be written as

$$|X| = x_{21}(-1)^{2+1}M_{21} + x_{22}(-1)^{2+2}M_{22} + x_{23}(-1)^{2+3}M_{23}$$

$$= -x_{21}\begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix} + x_{22}\begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} - x_{23}\begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}$$

$$= -x_{21}(x_{12}x_{33} - x_{13}x_{32}) + x_{22}(x_{11}x_{33} - x_{13}x_{31}) - x_{23}(x_{11}x_{32} - x_{12}x_{31})$$

$$= -x_{21}x_{12}x_{33} + x_{21}x_{13}x_{32} + x_{22}x_{11}x_{33} - x_{22}x_{13}x_{31} - x_{23}x_{11}x_{32} + x_{23}x_{12}x_{31}$$

$$= x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} - x_{13}x_{22}x_{31} \longrightarrow \text{(vi)}$$

Similarly, we can find $|X|$ for other values of i and j.

Note

If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

For example,

$$\Delta = a_{11}\,A_{21} + a_{12}\,A_{22} + a_{13}\,A_{23}$$
$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

Similarly, we can try for other rows and columns.

Example 2.10 Find det (A) if $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 6 & 2 \\ 8 & 0 & 1 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 6 & 2 \\ 8 & 0 & 1 \end{bmatrix}$

$$det(A) = |A| = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 6 & 2 \\ 8 & 0 & 1 \end{vmatrix} = 4\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 8 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 6 \\ 8 & 0 \end{vmatrix}$$

$$= 4(6-0) - 3(1-16) + 2(0-48)$$

$$= 24 + 45 - 96 = -27$$

Example 2.11 Evaluate $|X|$ by expanding across the third row.

$$X=\begin{bmatrix}3 & -1 & 2\\ 3 & 1 & 0\\ 1 & 0 & -1\end{bmatrix}$$

Solution: $|X|=(1)X_{31}+(0)X_{32}+(-1)X_{33}$

$$=(1)\,(-1)^{3+1}\begin{vmatrix}-1 & 2\\ 1 & 0\end{vmatrix}+(-1)\,(-1)^{3+3}\begin{vmatrix}3 & -1\\ 3 & 1\end{vmatrix}$$

$=-2-6=-8$

The value of this determinant is –8 no matter which row or column we expand upon.

2.3.4 Singular and Non-Singular Matrices.

A square matrix A is said to be **singular** if $|A|=0$, and **non-singular** if $|A|\neq 0$.

For example,

If $A=\begin{bmatrix}-9 & 3\\ -3 & -1\end{bmatrix}$ *and* $B=\begin{bmatrix}-1 & -2\\ 5 & 3\end{bmatrix}$

then $|A|=\begin{vmatrix}9 & 3\\ -3 & -1\end{vmatrix}=(9)(-1)-(3)(-3)=-9+9=0$

and $|B|=\begin{vmatrix}-1 & -2\\ 5 & 3\end{vmatrix}=(-1)\,(3)-(-2)\,(5)=-3+10=7.$

Since,

$|A|=0$, so the matrix $A=\begin{bmatrix}9 & 3\\ -3 & -1\end{bmatrix}$ is singular and $|B|$ $\neq 0$, so the matrix

$B=\begin{bmatrix}-1 & -2\\ 5 & 3\end{bmatrix}$ is non-singular.

2.3.5 Adjoint of *a* Square Matrix.

Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is a 2 × 2 square matrix. The **adjoint** of A denoted by *adj A*, it is defined to be the matrix whose elements are obtained by interchanging the places of a_{11} and a_{22} but by changing the signs of a_{12} and a_{21} that is $adj\ A=\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}_{2\times 2}$.

For example, if $A=\begin{bmatrix}6 & -8\\ 4 & -7\end{bmatrix}_{2\times 2}$ then adj $A=$ $\begin{bmatrix}-7 & 8\\ -4 & 6\end{bmatrix}_{2\times 2}$.

Now, let X be a square matrix of order 3 × 3 . Let X' denote the matrix obtained by replacing each element of X by its corresponding co-factor and taken transpose. Then X' is called the adjoint of X and is usually denoted by *adj X*.

$$X=\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix}$$

$$\text{Cofactors of } X=Y=\begin{bmatrix}X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{bmatrix}$$

$$adj\ X=Y^t=\begin{bmatrix}X_{11} & X_{21} & X_{31}\\ X_{12} & X_{22} & X_{32}\\ X_{13} & X_{23} & X_{33}\end{bmatrix}$$

Example 2.12 Calculate the adjoint of $X=$ $\begin{bmatrix}1 & -1 & 2\\ 2 & 3 & 5\\ 1 & 0 & 3\end{bmatrix}$.

Solution: Now the cofactors of the elements of X are

$$X_{11}=(-1)^{1+1}\begin{vmatrix}3 & 5\\ 0 & 3\end{vmatrix}=9$$

$$X_{12}=(-1)^{1+2}\begin{vmatrix}2 & 5\\ 1 & 3\end{vmatrix}=-1$$

$$X_{13}=(-1)^{1+3}\begin{vmatrix}2 & 3\\ 1 & 0\end{vmatrix}=-3$$

$$X_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 2\\ 0 & 3\end{vmatrix}=3$$

$$X_{22}=(-1)^{2+2}\begin{vmatrix}1 & 2\\ 1 & 3\end{vmatrix}=1$$

$$X_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$$

$$X_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11$$

$$X_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1$$

$$X_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

The cofactor matrix of $X = \begin{bmatrix} 9 & -1 & -3 \\ 3 & 1 & -1 \\ -11 & -1 & 5 \end{bmatrix}$

Therefore, adj $X = \begin{bmatrix} 9 & 3 & -11 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix}$

2.4 Properties of Determinants

We shall state some of the more obvious and useful properties of determinants which simplify the evaluation of determinants.

P-1. If every element in a row or column of a square matrix X is zero, then $|X| = 0$.

If $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ and every element in the first row is zero, then

$$X = \begin{bmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Now $|X| = x_{11}X_{11} + x_{12}X_{12} + x_{13}X_{13}$

$= 0X_{11} + 0X_{12} + 0X_{13} = 0$

We get the same result if every element of any other row or column is zero.

P-2. If any two rows or two columns in a square matrix A are interchanged, then the determinant of the resulting matrix is $-|A|$.

If

$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ *and* $Y = \begin{bmatrix} x_{21} & x_{22} & x_{23} \\ x_{11} & x_{12} & x_{13} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ is the matrix obtained by interchanging the first and second row of X, then

$$|Y| = \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{11} & x_{12} & x_{13} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

$= x_{21}(x_{12}x_{33} - x_{13}x_{32}) - x_{22}(x_{11}x_{33} - x_{13}x_{31})$
$\quad + x_{23}(x_{11}x_{32} - x_{21}x_{31})$

$= x_{21}x_{12}x_{33} - x_{21}x_{13}x_{32} - x_{22}x_{11}x_{33} + x_{22}x_{13}x_{31} + x_{23}x_{11}x_{32}$
$\quad - x_{23}x_{12}x_{31}$

$= -(x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32}$
$\quad - x_{13}x_{22}x_{31}) = -|X|.$

P-3. If *a* square matrix X has two identical rows or two identical columns, then $|X| = 0$

If $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ *and* $Y = \begin{bmatrix} x_{21} & x_{22} & x_{23} \\ x_{11} & x_{12} & x_{13} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ are the matrix

obtained by interchanging the first and second rows of X. Then by property (3), $|Y| = -|X|$. But the first and second rows of X are identical, mean $X = Y$ and so $|X| = |Y|$. Hence $|X| = -|X|$ or $2|X| = 0$ or $|X| = 0$. The same result is obtained if any two columns are identical.

P-4. Scalar Multiple Property

If every element of a row or column of a square matrix X is multiplied by the real number k, then the determinant of the resulting matrix is $k|X|$.

If $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ *and* $Y = \begin{bmatrix} kx_{11} & kx_{12} & kx_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

is the matrix obtained by multiplying first row of X by k. Then

$$|Y|=\begin{vmatrix} kx_{11} & kx_{12} & kx_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \begin{aligned} &= kx_{11}X_{11}+kx_{12}X_{12}+kx_{13}X_{13} \\ &= k(x_{11}X_{11}+x_{12}X_{12}+x_{13}X_{13}) \\ &= k|X| \end{aligned}$$

A similar result is obtained if any other row or column is multiplied by k and $k\in R$.

P-5. Sum Property

If every element of a row or column of a square matrix X is the sum of two terms, then its determinant can be written as the sum of two separate determinants.

If $$|X|=\begin{vmatrix} x_{11}+y_{11} & x_{12} & x_{13} \\ x_{21}+y_{21} & x_{22} & x_{23} \\ x_{31}+y_{31} & x_{32} & x_{33} \end{vmatrix}$$

Expanding by the first column, we have

$$|X|=(x_{11}+y_{11})X_{11}+(x_{21}+y_{21})X_{21}+(x_{31}+y_{31})X_{31}$$
$$=(x_{11}X_{11}+x_{21}X_{21}+x_{31}X_{31})+(y_{11}X_{11}+y_{21}X_{21}+y_{31}X_{31})$$

$$=\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}+\begin{vmatrix} y_{11} & x_{12} & x_{13} \\ y_{21} & x_{22} & x_{23} \\ y_{31} & x_{32} & x_{33} \end{vmatrix}$$

P-6. Property of Invariance

If every element of any row or column of a square matrix is multiplied by a real number k and the resulting product is added to the corresponding elements of another row or column of the matrix, then the determinant of the resulting matrix is equal to the determinant of the original matrix.

$$R_i \to R_i + (k)R_j$$

OR

$$C_i \to C_i + (k)C_j$$

If

$$X=\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \textit{then, } Y=\begin{bmatrix} x_{11}+kx_{12} & x_{12} & x_{13} \\ x_{21}+kx_{22} & x_{22} & x_{23} \\ x_{31}+kx_{32} & x_{32} & x_{33} \end{bmatrix} \text{is}$$

the Matrix obtained by multiplying k with every element of the second column of X and then adding to the corresponding element of the first column of X, then

$$|Y|=\begin{vmatrix} x_{11}+kx_{12} & x_{12} & x_{13} \\ x_{21}+kx_{22} & x_{22} & x_{23} \\ x_{31}+kx_{32} & x_{32} & x_{33} \end{vmatrix}$$

$$=\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}+\begin{vmatrix} kx_{12} & x_{12} & x_{13} \\ kx_{22} & x_{22} & x_{23} \\ kx_{32} & x_{32} & x_{33} \end{vmatrix} \quad \text{(by property (5))}$$

$$=\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}+k\begin{vmatrix} x_{12} & x_{12} & x_{13} \\ x_{22} & x_{22} & x_{23} \\ x_{32} & x_{32} & x_{33} \end{vmatrix} \text{(by property 4))}$$

$$=\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}+k(0) \qquad \text{(by property (3))}$$

$$=\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}=|X|.$$

P7. Triangle Property

This property of the determinant states that if the elements above or below the main diagonal are zero, then the value of the determinant is equal to the product of the diagonal elements. It means for any type of diagonal or scalar matrix, determinant is simply the product of its diagonal elements.

For any square matrix X such that,

$$|X|=\begin{vmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

$$|X|=x_{11}\times x_{22}\times x_{33}$$

P8. Transpose of Determinant (Reflection Property)

Transpose refers to the operations of interchanging rows and columns of the determinant. The rows become columns and columns become rows in order. It is denoted by $|X^t|$, for any determinant $|X|$. The property says determinant remains unchanged on its transpose, that is, $|\boldsymbol{X}^t| = |\boldsymbol{X}|$.

$$X=\begin{bmatrix} x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix} \; then, \; X^t=\begin{bmatrix} x_{11} & x_{21} & x_{31}\\ x_{12} & x_{22} & x_{32}\\ x_{13} & x_{23} & x_{33}\end{bmatrix}$$

$$X_{11}=(-1)^{1+1}\begin{vmatrix} x_{22} & x_{23}\\ x_{32} & x_{33}\end{vmatrix} \qquad X_{11}=(-1)^{1+1}\begin{vmatrix} x_{22} & x_{32}\\ x_{23} & x_{33}\end{vmatrix}$$

Hence, $X_{11}=X_{11}$ and $|X|=|X^t|$.

2.4.1 Inverse of a Square Matrix.

In this section, we'll build on your understanding of inverses from previous classes by exploring how to find and use the inverse of a square matrix, which plays a crucial role in solving matrix equations and applying these concepts to real-world problems.

A. Inverse of a 2 × 2 matrix

Inverse of a Matrix

For an $n \times n$ matrix **A**, if there is a matrix $\mathbf{A^{-1}}$ for which $\mathbf{A^{-1}\ A = I = A\ A^{-1}}$, then $\mathbf{A^{-1}}$ is the **inverse** of **A**.

From our previous class knowledge,

A $=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ is an non-singular matrix, then the inverse of A may be found by the formula

$$A^{-1}=\frac{1}{|A|}adj\ A$$

Note

(i) If A is non-singular, then A has an inverse.

(ii) Only square matrices possess an inverse.

(iii) If a matrix A has an inverse, then $A^{-1}A = AA^{-1} = 1$

Example 2.13 If $A=\begin{bmatrix} 3 & -4\\ 1 & -2\end{bmatrix}$, find A^{-1} and show $AA^{-1}=$ I.

Solution:

$$|A|=\begin{vmatrix} 3 & -4\\ 1 & -2\end{vmatrix}=(3)(-2)-(1)(-4)=-6+4=-2\neq 0,$$

so A is non-singular and thus A^{-1} exists.

By above formula, we have $A^{-1}=\dfrac{1}{|A|}adjA$, so

$$A^{-1}=\frac{1}{-2}\begin{bmatrix} -2 & 4\\ -1 & 3\end{bmatrix}=\begin{bmatrix} 1 & -2\\ \frac{1}{2} & \frac{-3}{2}\end{bmatrix}$$

Now $A^{-1}A$

$$=\frac{1}{-2}\begin{bmatrix} -2 & 4\\ -1 & 3\end{bmatrix}\begin{bmatrix} 3 & -4\\ 1 & -2\end{bmatrix}=-\frac{1}{2}\begin{bmatrix} -2 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}=\mathrm{I}$$

Thus $A^{-1}A=I.$

Example 2.14 If $A=\begin{bmatrix} 3 & -4\\ 1 & -2\end{bmatrix}$, Show that $(A^{-1})^{-1}=A$.

Solution: By above example, the inverse of A is given by

$$A^{-1}=\frac{1}{-2}\begin{bmatrix} -2 & 4\\ -1 & 3\end{bmatrix}=\begin{bmatrix} 1 & -2\\ \frac{1}{2} & \frac{-3}{2}\end{bmatrix}$$

Since $|A^{-1}|=$

$$\begin{vmatrix} 1 & -2\\ \frac{1}{2} & \frac{-3}{2}\end{vmatrix}=(1)\left(\frac{-3}{2}\right)-\left(\frac{1}{2}\right)(-2)=\frac{-3}{2}+1=-\frac{1}{2}\neq 0$$

we can find $(A^{-1})^{-1}$.

We have

$$\left(A^{-1}\right)^{-1}=\frac{1}{|A^{-1}|}adj\ A^{-1}=\frac{1}{-\frac{1}{2}}\begin{bmatrix} \frac{-3}{2} & 2\\ -\frac{1}{2} & 1\end{bmatrix}$$

$$=-2\begin{bmatrix} \frac{-3}{2} & 2\\ -\frac{1}{2} & 1\end{bmatrix}=\begin{bmatrix} 3 & -4\\ 1 & -2\end{bmatrix}=A$$

Thus $\left(A^{-1}\right)^{-1}=A$

B. Inverse of 3×3 matrix

Now, Let A be a square matrix of order 3 × 3. If there exists a square matrix Y of order n such that $XY = YX = I_n$ where I_n is the multiplicative identity matrix of order n, then Y is called the **multiplicative inverse** of X and is denoted by X^{-1}. Thus $XX^{-1}=X^{-1}X=I_n$.

It may be noted that inverse of a square matrix, if it exists, is unique. Moreover, if X is a non-singular square matrix of order n, then

$$X^{-1} = \frac{1}{|X|}\, adj\, X.$$

Example 2.15 Let $X = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$. Find X^{-1}

Solution: Since $X^{-1} = \frac{1}{|X|}\, adj\, X$, we need to find $adj\, X$ and $|X|$. First we find cofactor of every element of X.

$$X_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = (1)(0+1) = 1$$

$$X_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ -2 & 0 \end{vmatrix} = (-1)(0-2) = 2$$

$$X_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} = (1)(0+4) = 4$$

$$X_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = (-1)(0-2) = 2$$

$$X_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = (1)(0+4) = 4$$

$$X_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} = (-1)(2-2) = 0$$

$$X_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = (1)(1-4) = -3$$

$$X_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} = (-1)(-2-0) = 2$$

$$X_{33} = (-\ )^{3+3}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = (1)(4+0) = 4$$

So, $\quad adj\, X = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 2 \\ 4 & 0 & 4 \end{bmatrix}$

Next, we find $|X|$.

Since $\quad X = x_{11}X_{11} + x_{12}X_{12} + x_{13}X_{13}$

$= (2)(1) + (-1)(2) + (2)(4)$

$= 2 - 2 + 8$

$= 8 \neq 0$

Thus, $\quad X^{-1} = \frac{1}{|X|}\, adj\, X = \frac{1}{8}\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 2 \\ 4 & 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{-3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Skill 2.2

Matrix Operations Proficiency:

- Verify the transpose of matrix products.
- Calculate determinants, minors, and cofactors for square matrices.
- Recognize singular and non-singular matrices' significance.

Understanding Properties of Determinants:

- Apply and prove properties of determinants for various matrix operations.
- Evaluate determinants without expansion using determinant properties.

Exercise 2.2

1. *If* $A = \begin{bmatrix} 7 & -4 & 4 \\ 2 & 0 & -3 \\ -1 & 2 & -5 \end{bmatrix}$ *.then*

(i) Find M_{13}, M_{31} and M_{23}.

(ii) Find A_{11}, A_{32} and A_{23}.

(iii) Evaluate $|A|$ by expanding across the second row.

(iv) Evaluate $|A|$ by expanding down the second column.

2. Without evaluating state the reason for the following equalities.

(i) $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{bmatrix} = 0$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ -8 & 4 & -12 \\ 2 & -1 & 3 \end{bmatrix} = 0$

(iii) $\begin{vmatrix} 1 & 3 & -2 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 4 \end{vmatrix}$

(iv) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 2 & 4 & -6 \end{vmatrix} = -3\begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$

(vi) $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 5 & 5 & 6 \\ 1 & 2 & 2 \end{vmatrix}$

3. Evaluate the following determinants:

(i) $\begin{bmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 4 & -2 \\ 2 & 4 & -6 \\ -4 & 2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & -5 & 4 \\ -9 & 8 & -7 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & 0 \\ -2 & 3 & 4 \end{bmatrix}$

(v) $\begin{bmatrix} 3860 & 3861 \\ 3862 & 3863 \end{bmatrix}$ (vi) $\begin{bmatrix} 81 & 82 & 83 \\ 84 & 85 & 86 \\ 87 & 88 & 89 \end{bmatrix}$

4. Show that

(i) $\begin{vmatrix} a & b & c \\ l & m & n \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix}$

(ii) $\begin{vmatrix} a & b & c \\ 1-3a & 2-3b & 3-3c \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$

(iv) $\begin{vmatrix} bc & ca & ab \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

5. Identify singular and non-singular matrices.

(i) $\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iii) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^3 \end{vmatrix} = 4a^2b^2c^2$

(iv) $\begin{vmatrix} 1+x & y & z \\ x & 1+y & z \\ x & y & 1+z \end{vmatrix} = 1+x+y+z$

(v) $\begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = (x-p)(x-q)(x+p+q)$

(vi) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

6. Find x if the matrix $\begin{vmatrix} 1 & 2 & 1 \\ 0 & x & 1 \\ 1 & 2 & 6 \end{vmatrix}$ is singular.

7. Identify singular and non-singular matrices.

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & -3 \\ 1 & -1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 5 \\ 6 & 2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 0 & 6 & 2 \end{bmatrix}$ **(iv)** $A = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

8. Let $A = \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}$. Verify that $\left(A^{-1}\right)^t = \left(A^t\right)^{-1}$

9. Solve for x.

(i) $\begin{vmatrix} x+2 & 3 & 4 \\ 2 & x+2 & 4 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} -1 & 0 & 1 \\ x^2 & 1 & x \\ 2 & 3 & 4 \end{vmatrix} = 0$

10. Show that if inverse of a square matrix exists, then it is unique.

11. Let A = $\begin{bmatrix} 0 & 3 & 4 \\ -4 & 2 & 6 \\ 2 & 0 & 5 \end{bmatrix}$ Find A^{-1}.

12. Verify that $(AB)^t = B^t A^t$ if

(i) $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & i \\ -0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

13. Determine whether B is the inverse of A.

(i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} -2 & 0 & 3 \\ 5 & 1 & 7 \\ -3 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & -3 \\ 1 & 1 & 1 \\ -3 & 0 & 2 \end{bmatrix}$

14. Find the inverse of each of the following matrix.

(i) $\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

15. Verify that $(AB)^{-1} = B^{-1} A^{-1}$ if

$A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 4 \\ 4 & 3 \end{bmatrix}$

16. If A and B are non-singular matrices, then show that

(i) $\left(A^{-1}\right)^{-1} = A$ **(ii)** $(AB)^{-1} = B^{-1} A^{-1}$

17. If A = $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then verify that

$AA^{-1} = A^{-1}A = I_2$.

Student Learning Outcomes

✧ Use row operations to find the inverse and the rank of a matrix

2.5 Row and Column Operations

Building on your existing knowledge of matrices, this chapter introduces row and column operations—key techniques for transforming matrices to solve algebraic equations. We'll explore how swapping, scaling, and adding rows or columns simplify matrices and prepare them for further analysis.

2.5.1 Row operations on matrices.

In this section we explore essential matrix manipulation techniques used in linear algebra, such as row swapping, scaling, and addition. These operations are crucial for simplifying matrices, calculating determinants, and solving linear equations efficiently.

A. Elementary Row operations:

Any one of the following three operations performed on matrices are called elementary row operations:

(i) Interchange any two rows of the matrix.

(ii) Multiply every entry of some row of the matrix by the some non-zero scalar.

(iii) Add a multiple of one row of the matrix to another row.

We use the following representations to express the elementary row operations (i), (ii) and (iii):

- Interchanging of row R_i and R_j is represented by $R_i \leftrightarrow R_j$.
- Multiplication of a row R_i by a non-zero scalar k is denoted by kR_i.
- Adding k times R_i to R_j is expressed as $R_j + kR_i$.

Example 2.16 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 0 \\ -1 & -4 & 9 \end{bmatrix}$ perform the following elementary row operations on A.

(i) $R_3 \leftrightarrow R_1$

(ii) $R_1 - 4R_3$

Solution: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 0 \\ -1 & -4 & 9 \end{bmatrix}$

(i) $R_3 \leftrightarrow R_1$

$$= \begin{bmatrix} -1 & -4 & 9 \\ 3 & 5 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

(ii) $R_1 - 4R_3$

$$= \begin{bmatrix} 1-4(-1) & 2-4(-4) & 3+4(9) \\ 3 & 5 & 0 \\ -1 & -4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 18 & -33 \\ 3 & 5 & 0 \\ -1 & -4 & 9 \end{bmatrix}$$

2.5.2 Column operations on matrices.

Column operations on matrices are analogous to row operations, but they are performed exclusively on the columns of a matrix. These operations are useful for various purposes, such as simplifying matrices to a more manageable form in solving systems of linear equations, determining rank, and facilitating the calculation of determinants.

Example 2.17

Perform the following elementary column operations on $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 0 \\ -1 & -4 & 9 \end{bmatrix}$.

(i) $C_1 \leftrightarrow C_2$ **(ii)** $C_2 - C_1$

(iii) $C_1 \leftrightarrow C_2$: $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 3 & 0 \\ -4 & -1 & 9 \end{bmatrix}$

(iv) $C_2 - C_1$: $\begin{bmatrix} 1 & 2-1 & 3 \\ 3 & 5-3 & 0 \\ -1 & -4+1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 2 & 0 \\ -1 & -3 & 9 \end{bmatrix}$

2.5.3 Echelon and Reduced Echelon Forms of Matrices.

In this section, we'll learn about echelon and reduced echelon forms of matrices, which help us solve systems of equations and understand the structure of matrices. We'll see how to use simple row operations to simplify matrices and why these forms are useful in math problems.

A. Gauss Elimination Method [Echelon Form of a Matrix.]

An $m \times n$ matrix X is said to be in (row) echelon form (or an echelon matrix) if it satisfies the following properties.

(i) In each successive non-zero row, the number of zeros before the leading non-zero entry of a row increases row by row,

(ii) Every non-zero row in X proceeds every zero row (if there is any).

For example,

The matrices $\begin{bmatrix} 9 & 7 & -2 & 1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 6 & 8 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}$ are in echelon form, But the matrices $\begin{bmatrix} 0 & 0 & 1 & 6 \\ 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 8 & -13 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are not in echelon form.

B. Gauss Jordan Method [Reduced Echelon Form of a Matrix.]

An $m \times n$ matrix X is said to be in reduced (row) echelon form (or reduced echelon matrix) if it satisfies the following properties:

(i) It is in (row) echelon form,

(ii) The first non-zero entry in R_i lies in C_j is 1 and all other entries of C_j are zero.

For example, the matrices $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 8 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are in (row) reduced echelon form but $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not in (row) reduced echelon form.

2.5.4 Reduce a matrix to its echelon and reduced echelon form

The procedure to reduce a matrix to its echelon and reduced echelon form is illustrated in the following example.

Example 2.18 Reduce $A = \begin{bmatrix} 3 & 1 & -5 \\ 2 & 1 & -1 \\ 1 & -2 & -5 \end{bmatrix}$ to echelon form and then to reduced echelon form.

Solution:

$$\begin{bmatrix} 3 & 1 & -5 \\ 2 & 1 & -1 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{5}{3} \\ 2 & 1 & -1 \\ -1 & -2 & -5 \end{bmatrix} \text{ by } \frac{1}{3} R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & \frac{1}{3} & \frac{7}{3} \\ -1 & -2 & -5 \end{bmatrix} \text{ by } R_2 - 2R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & 7 \\ 0 & -\frac{5}{3} & -\frac{20}{3} \end{bmatrix} \text{ by } 3R_2 \text{ and } R_3 + R_1$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & 7 \\ 0 & -1 & -4 \end{bmatrix} \text{ by } \frac{3}{5} R_3$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & 7 \\ 0 & 0 & 3 \end{bmatrix} \text{ by } R_3 + R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & \frac{1}{3} & \frac{-5}{3} \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \text{ by } \frac{1}{3} R_3 \longrightarrow \text{(i)}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \text{ by } R_1 - \frac{1}{3} R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 7\\ 0 & 0 & 1\end{bmatrix} \text{ by } R_1 + 4R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ by } R_2 - 7R_3 \quad \longrightarrow \text{(ii)}$$

The matrices (i) and (ii) are in echelon form and reduced echelon form of the given matrix A respectively.

2.5.5 Inverse and rank of matrices by means of reduced echelon form.

A. Inverse of a Matrix:

Let A be a non-singular matrix. If we perform successive elementary row operations on the matrix $[A \mid I]$, which reduced A to I and I to the resulting matrix A^{-1} i.e. if $[A \mid I]$ is reduced to $[I \mid A^{-1}]$, then A^{-1} is the inverse of A.

Similarly, if we perform successive elementary column operation on the matrix $[A \mid I]$, which reduces A to I and I to the resulting matrix A^{-1}, then A^{-1} is the inverse of A.

Example 2.19 Find the inverse of the matrix

$$X = \begin{bmatrix}3 & 2 & 1\\ 4 & 5 & 2\\ -1 & 2 & -2\end{bmatrix}$$

Solution: Since

$$\begin{vmatrix}3 & 2 & 1\\ 4 & 5 & 2\\ -1 & 2 & -2\end{vmatrix} = 3\begin{vmatrix}5 & 2\\ 2 & -2\end{vmatrix} - 2\begin{vmatrix}4 & 2\\ -1 & -2\end{vmatrix} + 1\begin{vmatrix}4 & 5\\ -1 & 2\end{vmatrix}$$

Expanding by 1st row

$$= 3(-10-4) - 2(-8+2) + 1(8+5)$$

$$= -42 + 12 + 13$$

$$= -17 \neq 0$$

So, matrix X is non-singular and its inverse X^{-1} exists.

Now, $$\left[\begin{array}{ccc|ccc}3 & 2 & 1 & 1 & 0 & 0\\ 4 & 5 & 2 & 0 & 1 & 0\\ -1 & 2 & -2 & 0 & 0 & 1\end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}-1 & 2 & -2 & 0 & 0 & 1\\ 4 & 5 & 2 & 0 & 1 & 0\\ 3 & 2 & 1 & 1 & 0 & 0\end{array}\right] R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 0 & 0 & -1\\ 4 & 5 & 2 & 0 & 1 & 0\\ 3 & 2 & 1 & 1 & 0 & 0\end{array}\right] \text{by } -R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 0 & 0 & -1\\ 0 & 13 & -6 & 0 & 1 & 4\\ 0 & 8 & -5 & 1 & 0 & 3\end{array}\right] \text{by } R_2 - 4R_1 \text{ and } R_3 - 3R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 0 & 0 & -1\\ 0 & 1 & \frac{-6}{13} & 0 & \frac{1}{13} & \frac{4}{13}\\ 0 & 8 & -5 & 1 & 0 & 3\end{array}\right] \text{by } \frac{1}{13} R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 1 & 0 & -1\\ 0 & 1 & \frac{-6}{13} & 0 & \frac{1}{13} & \frac{4}{13}\\ 0 & 0 & \frac{-17}{13} & \frac{1}{8} & \frac{-8}{13} & \frac{7}{13}\end{array}\right] \text{by } R_3 - 8R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 1 & 0 & -1\\ 0 & 1 & \frac{-6}{13} & 0 & \frac{1}{13} & \frac{4}{13}\\ 0 & 0 & -17 & -13 & 8 & -7\end{array}\right] \text{by } -13R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc}1 & -2 & 2 & 1 & 0 & -1\\ 0 & 1 & \frac{-6}{13} & 0 & \frac{1}{13} & \frac{4}{13}\\ 0 & 0 & 1 & \frac{-13}{17} & \frac{8}{17} & \frac{-7}{17}\end{array}\right] \text{by } \frac{1}{17} R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{14}{13} & 0 & \frac{2}{13} & -\frac{5}{13} \\ 0 & 1 & \frac{-6}{13} & 0 & \frac{1}{13} & \frac{4}{13} \\ 0 & 0 & 1 & -\frac{13}{17} & \frac{8}{17} & -\frac{7}{17} \end{array}\right] \text{by } R_1+2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{14}{17} & -\frac{6}{17} & \frac{1}{17} \\ 0 & 1 & 0 & -\frac{6}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & -\frac{13}{17} & \frac{8}{17} & -\frac{7}{17} \end{array}\right] \begin{array}{l} \text{by } R_2+\frac{6}{13}R_3 \\ \text{and } R_1-\frac{14}{13}R_3 \end{array}$$

Thus, The inverse of $X = X^{-1} = \begin{bmatrix} \frac{14}{17} & -\frac{6}{17} & \frac{1}{17} \\ -\frac{6}{17} & \frac{5}{17} & \frac{2}{17} \\ -\frac{13}{17} & \frac{8}{17} & -\frac{7}{17} \end{bmatrix}$

B. Rank of a Matrix

Let X be a non-zero matrix. The rank of an $m \times n$ matrix X denoted by "rank (X") is defined to be the number of non-zero rows in the row echelon form of X.

Example 2.20 Find the rank of $X = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

Solution: Since, $X = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \text{by } R_2-2R_1 \text{ and } R_4-R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \text{by } -R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{by } R_3-2R_2 \text{ and } R_4-R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{by } \frac{1}{3}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{by } R_4-4R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{by } R_2+R_3 \text{ and } R_1-R_3$$

The number of non-zero rows of last matrix are three so, the "rank of X" is 3.

Skill 2.3

Matrix Row Operations Mastery:

- Perform row and column operations on matrices proficiently.
- Reduce matrices to echelon and reduced echelon forms effectively.

Exercise 2.3

1. Reduce each of the following matrices to the indicated form Echelon form

(i) $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & 9 \\ 5 & 8 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & -3 \\ 3 & 2 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 5 & 3 \\ 5 & -4 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

2. Reduce each of the following matrices to the indicated form Reduced Echelon form

(i) $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 3 \\ 3 & -4 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 8 & -1 \\ 1 & 2 & 3 \\ 5 & -5 & 1 \\ 4 & 1 & 9 \end{bmatrix}$

3. Find the inverses of the following matrices by using elementary row and column operations.

(i) $\begin{bmatrix} 3 & -2 & 2 \\ 2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -1 & 7 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -7 & 6 \\ 2 & 1 & 0 \\ 1 & -3 & 5 \end{bmatrix}$

4. Find the ranks of each of the following matrices.

(i) $\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 & -4 \\ 0 & 2 & 1 \\ 2 & -1 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & 4 \\ 2 & 1 & -4 \\ -1 & 3 & 5 \end{bmatrix}$

Student Learning Outcomes

- ✧ Explain a consistent and inconsistent system of linear equations and demonstrate through examples
- ✧ Solve a system of 3 by 3 nonhomogeneous linear equations by using matrix inversion method and Cramer's Rule
- ✧ Solve a system of three homogeneous linear equations in three unknowns using the Gaussian elimination method

2.6 Solving System of Linear Equations

A system of equations is a collection of equations that we deal with altogether at once. A simple linear equations system consists of two equations and two variables. There are two kinds of equations.

(i) Homogeneous equations

(ii) Non-homogeneous equations

2.6.1 Homogeneous and non-homogeneous equations.

Consider the equation $a_1 x_1 + a_2 x_2 = k$ $\longrightarrow$ (i)

where a_1 and a_2 are not simultaneously zero. The equation (i) is known as a non-homogeneous linear equation in two variables (or unknowns) x_1 and x_1.

Now consider the following two non-homogeneous linear equations in two variables x_1 and x_2.

$$\left.\begin{aligned} a_1x_1 + a_2x_2 = k_1 \\ b_1x_1 + b_2x_2 = k_2 \end{aligned}\right\} \longrightarrow \text{(ii)}$$

If we take $k_1 = 0$ in equations (i), then it takes the form $a_1 x_1 + a_2 x_2 = 0$ $\longrightarrow$ (iii)

This form of linear equation is known as a homogeneous linear equation in two variables x_1 and x_2. If we take $k_1 = k_2 = 0$ in (ii), then

$$\left.\begin{aligned} a_1x_1 + a_2x_2 = 0 \\ b_1x_1 + b_2x_2 = 0 \end{aligned}\right\} \longrightarrow \text{(iv)}$$

is known as a system of homogeneous linear equations in the variables x_1 and x_2.

Similarly, the following equation,

$a_1x_1 + a_2x_2 + a_3x_3\ k_1$ where $a, b, c \neq 0$, and $k \neq 0 \longrightarrow$ (v)

is known as a **non-homogeneous linear equation** in three variables x_1, x_2 and x_3 and the following is a system of three non-homogeneous linear equations in three variables x_1,x_2 and x_3.

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 = k_1 \\ b_1x_1 + b_2x_2 + b_3x_3 = k_2 \\ c_1x_1 + c_2x_2 + c_3x_3 = k_3 \end{aligned}\right\} \longrightarrow \text{(vi)}$$

together form a system of non-homogeneous linear equations in three variables x_1, x_2 and x_3.

If we take $k_1 = 0$ in (v), then

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \longrightarrow \text{(vii)}$$

is known as a homogeneous equation in three variables x_1, x_2 and x_3.

If we take $k_1 = k_2 = k_3 = 0$ in (vi) then

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 = 0 \\ b_1x_1 + b_2x_2 + b_3x_3 = 0 \\ c_1x_1 + c_2x_2 + c_3x_3 = 0 \end{aligned}\right\} \longrightarrow \text{(viii)}$$

is called system of homogeneous linear equations in three variables x_1, x_2 and x_3.

An order triple (t_1, t_2, t_3) is called a **solution** of system (vi) if the equations are true for $x_1 = t_1$, $x_2 = t_2$ and $x_3 = t_3$. The **solution set** is denoted by $S = \{(t_1, t_2, t_3)\}$. In the case of system (viii), we see that it is always true for $x_1 = t_1 = 0$, $x_2 = t_2 = 0$ and $x_3 = t_3 = 0$, so the order triple $(t_1, t_2, t_3) = (0,0,0)$ is a solution of the system. Such a solution is called the **trivial** (or **zero**) **solution** and any other solution, if it exists, other than trivial solution is called a **non-trivial** (or **non-zero**) solution of the system.

Note

In writing the augmented matrix of a linear system enter zero (0) whenever a variable is missing in the equations, so, the coefficient of the variable is zero.

Consider system (vi). Since

$$\begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 \\ b_1x_1 + b_2x_2 + b_3x_3 \\ c_1x_1 + c_2x_2 + c_3x_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

the system (vi) may be written as a single matrix equation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \longrightarrow \text{(ix)}$$

or $\quad AX = B \longrightarrow$ (x)

where, $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$.

A is called the **matrix of coefficients**, X is the column vector of variables and B is the column vector of constants.

If we adjoin the column vector B of the constants to the matrix A on the right separated by a bar, that is

$[A|B] = \left[\begin{array}{ccc|c} a_1 & a_2 & c_3 & k_1 \\ b_2 & b_2 & b_3 & k_2 \\ c_3 & c_3 & c_3 & k_3 \end{array}\right]$, the new matrix so obtained is called **augmented matrix** of the given system.

2.6.2 Solve a system of three homogeneous linear equations in three unknowns

(i) Solution of homogeneous linear equations

Consider the following system of three homogeneous linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \quad (i) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \quad (ii) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \quad (iii) \end{aligned}\right\} \longrightarrow \text{(I)}$$

which is equivalent to the matrix equation

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or simply $AX = O$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and O $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

If $|A| \neq 0$, then A is non-singular and A^{-1} exists.

We have $A^{-1}(AX) = A^{-1}0 \Rightarrow (A^{-1}A)X = 0 \Rightarrow \mathrm{I}X = 0 \Rightarrow X = 0$, that is

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $x_1 = 0, x_2 = 0$ and $x_3 = 0$. This shows that the system has trivial solution. So, we may conclude

"A system $AX = 0$ of three homogeneous linear equations in three variables has a trivial solution if A is non-singular i.e. $|A| \neq 0$".

Next we find the condition under which the system (I) has a non–trivial solution. Multiplying equations (i), (ii) and (iii) of the system by the co–factors A_{11}, A_{21} and A_{31} of the corresponding elements a_{11}, a_{21} and a_{31} and then adding them up, we get

$$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})x_2 + (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})x_3 = 0$$

From this, we have $|A|x_1 = 0$. Likewise, we can have $|A|x_2 = 0$ and $|A|x_3 = 0$. The system (1) has a non-trivial solution if at least one of the variable x_1, x_2 and x_3 is different from zero. Suppose $x_1 \neq 0$, then $|A|\, x_1 = 0 \Rightarrow |A| = 0$. Thus, we may conclude:

"A system $AX = 0$ of three homogeneous linear equations in three variables has a non-trivial solution if it is singular i.e. $|A| = 0$"

Example 2.21 Show that the following system has a trivial solution.

$$2x_1 + 3x_2 - 3x_3 = 0 \longrightarrow \text{(i)}$$

$$x_1 + 2x_2 - 2x_3 = 0 \longrightarrow \text{(ii)}$$

$$x_1 + 2x_2 + 2x_3 = 0 \longrightarrow \text{(iii)}$$

Solution: Since

$$|A| = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 2 & -2 \\ 1 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$$

$= 2(4 + 4) - 3(2 + 2) - 3(2 - 2)$

$= 16 - 12 = 4 \neq 0$, the system has a trivial solution. Subtracting equation (iii) from (ii), we get $x_3 = 0$ by putting $x_3 = 0$ in (i) and (ii) then subtracting equation $2\times$(ii) from (i), we have $x_2 = 0$. Putting $x_2 = 0$ and $x_3 = 0$ in equation (i) we obtain $x_1 = 0$, and therefore $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and the system has only trivial solution.

Example 2.22 Show that the system ha non-trivial solution

$$x_1 + x_2 + x_3 = 0 \longrightarrow \text{(i)}$$

$$x_2 - x_3 = 0 \longrightarrow \text{(ii)}$$

$$x_1 + 2x_2 = 0 \longrightarrow \text{(iii)}$$

Solution: Since

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 0$, the system has a non-trivial solution. Using equation (ii), we get

$$x_2 = x_3 \longrightarrow \text{(iv)}$$

Using equation (iv) in (iii),

$$x_1 + 2x_3 = 0$$

$$x_1 = -2x_3$$

We get that $x_1 = -2t$, $x_2 = t$ and $x_3 = t$, where t is ant real number. Thus, the given system has infinitely many solutions.

Example 2.23 For what value of λ the system has a non-trivial solution. Solve the system for the value of λ.

$$x_1 + x_2 + 2x_3 = 0$$
$$2x_1 + x_2 + \lambda x_3 = 0$$
$$3x_1 + x_2 + 2x_3 = 0$$

Solution: First we find the value of λ. We have

$$A=\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & \lambda\\ 3 & 1 & 2\end{bmatrix},$$

So
$$|A|=\begin{vmatrix}1 & -1 & 2\\ 2 & 1 & \lambda\\ 3 & 1 & 2\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ 2 & 3 & \lambda-4\\ 3 & 4 & -4\end{vmatrix}$$

$$=1.\begin{vmatrix}3 & \lambda-4\\ 4 & -4\end{vmatrix}=-12-4(\lambda-4)=4-4\lambda.$$

We know that the system has non-trivial solution if $|A|$=0, that is $4-4\lambda$ =0 or λ =1. Substituting the value of λ into the system, we have

$$x_1+x_2+2x_3=0$$
$$2x_1+x_2+x_3=0$$
$$3x_1+x_2+2x_3=0$$

Now solving the first two equations, we get $x_1=-x_3$. Putting this value in the third equation, we obtain $-3x_3+x_2+2x_3=0$ which $x_2=x_3$. We see that $x_1=-t$, $x_2=t$ and $x_3=t$ satisfy all the three equations of the system for any real value of t. Thus, the given system has infinitely many solutions for λ =1.

2.6.3 Consistent and inconsistent system of linear equations

(i) A system of linear equations is said to be consistent if the system has only one (i.e. unique) solution or it has infinitely many solutions.

(ii) A system of linear equations is said to be inconsistent if the system has no solution.

A. Consistency criterion for a system

To find the criterion for a system of linear equations to be consistent or inconsistent, we consider the following three systems of linear equations in three variables.

$$\left\{\begin{aligned}&2x_1+2x_2-x_3=4\\ &x_1+x_2=0\\ &x_1-2x_2+x_3=2\end{aligned}\right\}\longrightarrow \text{(I)}$$

$$\left\{\begin{aligned}&-x_1-x_2+2x_3=1\\ &x_1-2x_2+x_3=2\\ &x_1-5x_2+4x_3=5\end{aligned}\right\}\longrightarrow \text{(II)}$$

$$\left\{\begin{aligned}&x_1-2x_2+3x_3=1\\ &-2x_1+5x_2-4x_3=-2\\ &x_1-4x_2-x_3=5\end{aligned}\right\}\longrightarrow \text{(III)}$$

We solve these systems now by performing the elementary row operations on the augmented matrices of these systems so that to reduce them to (row) echelon form.

1. Consider system (I). the augmented matrix of the systems is

$$\left[\begin{array}{ccc|c}2 & 2 & -1 & 4\\ 1 & 1 & 0 & 0\\ 1 & -2 & 1 & 2\end{array}\right]$$

then $\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 1 & 1 & 0 & 0\\ 2 & 2 & -1 & 4\end{array}\right]$ by $R_1\leftrightarrow R_3$

$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 3 & -1 & -2\\ 0 & 6 & -3 & 0\end{array}\right]$ by R_3-2R_1 and R_2-R_1

$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 1 & \frac{-1}{3} & \frac{-2}{3}\\ 0 & 6 & -3 & 0\end{array}\right]$ by $\frac{1}{3}R_2$

$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 1 & \frac{-1}{3} & \frac{-2}{3}\\ 0 & 0 & -1 & 4\end{array}\right]$ by R_3-6R_2

$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & -2 & 1 & 2\\ 0 & 1 & \frac{-1}{3} & \frac{-2}{3}\\ 0 & 0 & 1 & -4\end{array}\right]$ by $-R_3$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{-1}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & -4 \end{array}\right] \text{by } R_1 + 2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] \text{by } R_2 + \frac{1}{3}R_3 \text{ and } R_1 - \frac{1}{3}R_3$$

Thus, the solution of the system is $x_1 = 2$, $x_2 = -2$ and $x_3 = -4$. Since the system has a solution, so it is consistent.

Remember

Two systems of equations are said to be equivalent if they have the same solution set.

2. Consider system (II). The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 1 \\ 1 & -2 & 1 & 2 \\ 1 & -5 & 4 & 5 \end{array}\right]$$

then $\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & -5 & 4 & 5 \end{array}\right]$ by $R_1 \leftrightarrow R_2$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & -3 & 3 & 3 \end{array}\right] \text{by } R_2 + R_1 \text{ and } R_3 - R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{by } R_3 - R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{by } \frac{1}{3} R_2.$$

The system (II) is reduced to the equivalent system

$$x_1 - 2x_2 + x_3 = 2 \longrightarrow \text{(i)}$$

$$-x_2 + x_3 = 1 \longrightarrow \text{(ii)}$$

$$0x_3 = 0 \longrightarrow \text{(iii)}$$

Equation (iii) is obviously satisfied for all choices of x_3. Equations (i) and (ii) yield

$$x_1 = 2 + 2x_2 - x_3 \longrightarrow \text{(iv)}$$

$$x_2 = x_3 - 1 \longrightarrow \text{(v)}$$

Since x_3 is arbitrary, from equations (iv) and (v) we can find infinitely many values of x_1 and x_2. This is equivalent to saying that the system has infinitely many solutions. Thus the system is consistent.

3. Consider system (III). The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -2 & 5 & -4 & -2 \\ 1 & -4 & -1 & 5 \end{array}\right]$$

then $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -2 & 5 & -4 & -2 \\ 1 & -4 & -1 & 5 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 4 \end{array}\right]$ by $R_2 + R_1$ and $R_3 - R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right] \text{by } R_3 + 2R_2$$

The system (III) is reduced to the equivalent system

$$x_1 - 2x_2 + 3x_3 = 1 \longrightarrow \text{(i)}$$

$$x_2 + 2x_3 = 0 \longrightarrow \text{(ii)}$$

$$0x_3 = 4 \longrightarrow \text{(iii)}$$

We see that the equation (iii) has no solution. Therefore, this system of equations has no solution. Hence the system is inconsistent.

From the above, we note that system of linear equations may have no solution, have only one solution, or have infinitely many solutions.

2.6.4 Solution of System of 3 × 3 Non-homogeneous Linear Equations

A system of non-homogeneous linear equations can be solved by using the following methods:

(a) Matrix Inversion Method i.e. $AX=B \Rightarrow X=A^{-1}B$

(b) Gauss Elimination Method (Echelon Form)

(c) Gauss–Jordan Method (Reduced Echelon Form)

(d) Cramer's Rule.

(a) Matrix Inversion Method:

Consider the following system of three non-homogeneous linear equations in three variables x_1, x_2 and x_3.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{aligned}\right\}$$

This system is equivalent to the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \text{ or } AX = B, \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

If A is non-singular, then A^{-1} exists. So, we have

$$AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X$$
$$= A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B.$$

Therefore, the matrix of variables is now determined as the product of A^{-1}B.

The method discussed above for finding the solution of a system of non–homogenous linear equations is known as matrix inversion method.

Example 2.24 Solve the system of equations by matrix method inversion method.

$$3x_1 - 2x_2 + 5x_3 = 5$$
$$x_1 + x_2 - 3x_3 = 0$$
$$x_1 + x_3 = 7$$

Solution: In matrix form

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

Since, $$A^{-1} = \frac{1}{|A|} Adj\ A \longrightarrow \text{(i)}$$

So,

$$|A| = \begin{vmatrix} 3 & -2 & 5 \\ 1 & 1 & -3 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 5 \\ 1 & -3 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 1(6-5) - 0(-9-5) + 1(3+2)$$

$$= 1 - 0 + 5$$

$$= 6 \neq 0$$

So, A^{-1} exists.

$$Adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} = 1(1+0) = 1,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = -1(1+3) = -4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1(0-1) = -1,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 5 \\ 0 & 1 \end{vmatrix} = -1(-2-0) = 2,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = 1(3-5) = -2,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -1(0+1) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 5 \\ 1 & -3 \end{vmatrix} = 1(6-5) = 1,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -(-9-5) = 14$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 1(3+2) = 5$$

Now,

$$Adj\ A = \begin{bmatrix} 1 & 2 & 1 \\ -4 & -2 & 14 \\ -1 & -2 & 5 \end{bmatrix}$$

By putting the values in equation (i), we get

$$A^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{2}{6} & \frac{1}{6} \\ -\frac{4}{6} & -\frac{2}{6} & \frac{14}{6} \\ -\frac{1}{6} & -\frac{2}{6} & \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

But

$$X = A^{-1}B$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \times \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

That is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

Thus, $x_1 = 2, \ \ x_2 = 13, \ \ x_3 = 5$ which is the solution of the given system.

Note

The matrix inversion method for solving a system of non-homogeneous linear equations is applicable only when the co-efficient matrix A is non-singular i.e. $|A| \neq 0$.

(b) Gauss Elimination Method (Echelon Form)

We are already familiar with the method for reducing the augmented matrix of a system of non-homogeneous linear equations to echelon form. We now apply this method to find the solution of a system of non-homogeneous linear equations. The procedure is known as Gauss Elimination Method.

Example 2.25 Solve the following system by the Gauss Elimination method.

$$2x_1 + 2x_2 - x_3 = 4$$
$$x_1 + x_2 = 0$$
$$x_1 - 2x_2 + x_3 = 2$$

Solution: The augmented matrix of the given system is

$$\left[\begin{array}{ccc|c} 2 & 2 & -1 & 4 \\ 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 2 \end{array}\right]$$

Then $\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 4 \end{array}\right]$ By $R_1 \leftrightarrow R_3$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & 6 & -3 & 0 \end{array}\right]$ By $R_2 - R_1$ and $R_3 - 2R_1$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 6 & -3 & 0 \end{array}\right]$ By $\frac{1}{3}R_2$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & -\frac{1}{3} & | & -\frac{2}{3} \\ 0 & 0 & -1 & | & 4 \end{bmatrix} \text{By } R_3 - 6R_2$$

So, the equivalent system of echelon form is

$$x_1 - 2x_2 + x_3 = 2 \longrightarrow \text{(i)}$$

$$x_2 - \frac{1}{3}x_3 = -\frac{2}{3} \longrightarrow \text{(ii)}$$

$$-x_3 = 4 \longrightarrow \text{(iii)}$$

From the above equation (iii), $x_3 = -4$, putting this value to the equation (ii), we get $x_2 = -2$. Now, putting $x_2 = -2$ and $x_3 = -4$ in equation (i) to get $x_1 = 2$.

Thus $x_1 = 2$, $x_2 = -2$, $x_3 = -4$ is the solution of the given system.

(c) Gauss–Jordan Method (Reduced Echelon Form)

Consider system of equations in example 2.25 and the echelon form

$$\begin{bmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & -\frac{1}{3} & | & -\frac{2}{3} \\ 0 & 0 & -1 & | & 4 \end{bmatrix}$$

We reduce the above augmented matrix to reduced (row) echelon form, that is

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & -\frac{1}{3} & | & -\frac{2}{3} \\ 0 & 0 & 1 & | & -4 \end{bmatrix} \text{By} -R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{1}{3} & | & \frac{2}{3} \\ 0 & 1 & -\frac{1}{3} & | & -\frac{2}{3} \\ 0 & 0 & 1 & | & -4 \end{bmatrix} \text{By } R_1 + 2R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -4 \end{bmatrix} \text{By } R_2 + \frac{1}{3}R_3 \text{ and } R_1 - \frac{1}{3}R_3$$

The equivalent system in the reduced (row) echelon form is

$$x_1 = 2,\ x_2 = -2 \ \ and \ \ x_3 = -4$$

which is solution of the given system.

The procedure illustrated above of transforming a system of non–homogeneous linear equations into an equivalent system in the reduced (row) echelon form from which solutions are easy to obtained is called the Gauss–Jordan Method.

(d) Cramer's Rule

Consider the following system of three non-homogeneous linear equations in three variables.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{aligned}\right\} \longrightarrow \text{(i)}$$

which is equivalent to the matrix equation

$$AX = B \ \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $and\ B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

Thus

$$x_1 = \frac{k_1A_{11} + k_2A_{21} + k_3A_{31}}{|A|} = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$$

$$x_2 = \frac{k_1A_{12} + k_2A_{22} + k_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{|A|}$$

$$x_3 = \frac{k_1A_{13} + k_2A_{23} + k_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & k_{12} & k_1 \\ a_{21} & k_{22} & k_2 \\ a_{31} & k_{32} & k_3 \end{vmatrix}}{|A|}$$

This method of finding the solution of the system is called Cramer's Rule.

Interesting Information

The use of matrices in solving equations dates back to ancient China, where the mathematician Liu Hui used them in the 3rd century AD to solve systems of linear equations. The formal development of matrix theory began in the 19th century with the work of Arthur Cayley and James Joseph Sylvester. They introduced matrix notation and operations, which revolutionized mathematical computations. Today, matrices are fundamental in numerous fields, including engineering, physics, economics, and computer science, enabling efficient and systematic solutions to complex problems.

Note

(i) Like matrix method, the Cramer's Rule is also applicable only when $|A| \neq 0$

(ii) Cramer's Rule is simpler than matrix method for finding solution of the given system.

Example 2.26 Use Cramer's Rule to solve the following system.

$$2x_1 + x_2 - x_3 = 5$$
$$2x_1 + 3x_2 - x_3 = 0$$
$$x_1 + x_2 = 8$$

Solution: The above system of equation in matrix form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 2 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 0 \\ 8 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

$= 1(-1+3) - 1(-2+2) + 0(6-2)$

$= 2 - 0 + 0$

$|A| = 2$

Now,

$$x_1 = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{31} & a_{32} \end{vmatrix}}{|A|}$$

Now,

$$= \frac{\begin{vmatrix} 5 & 1 & -1 \\ 0 & 3 & -1 \\ 8 & 1 & 0 \end{vmatrix}}{2} \quad \textit{(Expanding with 3}^{rd}\textit{ row)}$$

$$= \frac{8\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 5 & -1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 5 & 1 \\ 0 & 3 \end{vmatrix}}{2}$$

$$= \frac{8(-1+3) - 1(-5+0) + 0}{2}$$

$$= \frac{16+5}{2} = \frac{21}{2} = 10\frac{1}{2}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{|A|}$$

$$= \frac{\begin{vmatrix} 2 & 5 & -1 \\ 2 & 0 & -1 \\ 1 & 8 & 0 \end{vmatrix}}{2} \quad \textit{(Expanding with 3}^{rd}\textit{ row)}$$

$$= \frac{1\begin{vmatrix} 5 & -1 \\ 0 & -1 \end{vmatrix} - 8\begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 5 \\ 2 & 0 \end{vmatrix}}{2}$$

$$= \frac{1(-5+0) - 8(-2+2) + 0(0-10)}{2} = -\frac{5}{2}$$

$$= -2\frac{1}{2}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{|A|}$$

$$= \frac{\begin{vmatrix} 2 & 1 & 5 \\ 2 & 3 & 0 \\ 1 & 1 & 8 \end{vmatrix}}{2} \quad \textit{(Expanding with } 3^{rd} \textit{ row)}$$

$$= \frac{1\begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 2 & 0 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}}{2}$$

$$= \frac{1(0-15)-1(0-10)+8(6-2)}{2}$$

$$= \frac{-15+10+32}{2}$$

$$= \frac{27}{2} = 13\frac{1}{2}$$

Therefore, $x_1 = 10\frac{1}{2}$, $x_2 = -2\frac{1}{2}$ *and* $x_3 = 13\frac{1}{2}$ is the solution of the given system.

Skill 2.4

Systems of Linear Equations Problem-Solving Skills:

- Differentiate between homogeneous and non-homogeneous equation systems.
- Identify consistent and inconsistent equation systems.
- Utilize multiple methodologies to solve equation systems efficiently.

Exercise 2.4

1. Solve the following systems of equations by matrix inversion method.

(i) $3x_1 - 6x_2 + x_3 = 12$
$5x_1 + x_2 - 4x_3 = -3$
$x_1 + 2x_2 - 2x_3 = 2$

(ii) $2x_1 + x_2 + x_3 = 4$
$x_1 + 3x_2 - 2x_3 = 3$
$2x_1 + x_2 + 2x_3 = 1$

(iii) $x_1 + 4x_2 + 3x_3 = 8$
$7x_1 + x_3 = 2$
$4x_2 + 3x_3 = 9$

2. Solve the following system of equations by the Gauss elimination method and Gauss–Jordan method.

(i) $3x - 3y + 4z = 0$
$x + y - 4z = 4$
$3x + 4y - 2z = -3$

(ii) $4x - 3y + 2z = 0$
$3x - 2y - 2z = 2$
$x + 2y + 52z = 7$

(iii) $2x_2 - x_3 = 7$
$x_1 + 3x_3 = 9$
$x_1 + x_2 - x_3 = 4$

3. Use Cramer's rule to solve the following system of equations.

(i) $x - 2y = -4$
$3x + y = -8$
$2x + z = -7$

(ii) $x - y + 2z = 10$
$2x + y - 2z = -3$
$4x + y + z = 8$

(iii) $2x_1 + 3x_2 + 4x_3 = 5$
$x_2 - 2x_3 = 7$
$4x_1 - x_2 = 8$

4. Solve the following system of homogeneous equations.

(i) $4x_1 - x_2 + x_3 = 0$
$3x_1 + 2x_2 - x_3 = 0$
$2x_1 + x_2 + 3x_3 = 0$

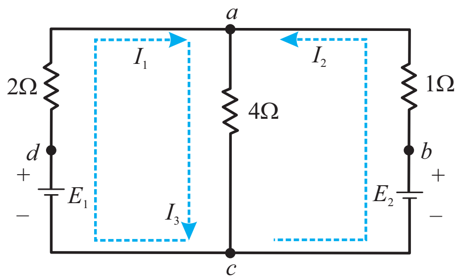
(ii) $x_1 + 6x_2 - 8x_3 = 0$
$-8x_1 + x_2 - 2x_3 = 0$
$-x_1 + 7x_2 + 3x_3 = 0$

5. For what value of λ the following system of homogeneous equations has a non-trivial solution. Solve the system.

$x_1 + 5x_2 + 3x_3 = 0$
$5x_1 + x_2 - \lambda x_3 = 0$
$x_1 + 2x_2 + \lambda x_3 = 0$

6. Circuit Analysis Consider the circuit shown in the figure. The currents I_1, I_2, and I_3, in amperes, are the solution of the system of linear equations.

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$



Where E_1 and E_2 are voltages. Use the Either Cramer's Rule or Matrix Inversion to find the unknown currents for the voltages E_1= 14 volts and E_2 = 28 volts.

Student Learning Outcomes

- Apply concepts of matrices to real world problems such as (graphic design, data encryption, seismic analysis, cryptography, transformation of geometric shapes, social network analysis)

2.7 Application of Matrices in Real Life

Matrices are not just abstract mathematical concepts; they are powerful tools used in various real-world applications. From the intricate designs in graphic arts to the complex algorithms in data encryption, matrices play a crucial role. They help in analyzing seismic data for predicting earthquakes, transforming geometric shapes in computer graphics, and even in understanding social networks. This section explores how the fundamental concepts of matrix multiplication, inverses, determinants, and Cramer's rule can be applied to solve practical problems in these diverse fields.

Use the information given in the table to set up a matrix to find the camera sales in each city.

Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means "hidden.") Matrix multiplication can be used to encode and decode messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0=__	9=I	18=R
1=A	10=J	19=S
2=B	11=K	20=T
3=C	12=L	21=U
4=D	13=M	22=V
5=E	14=N	23=W
6=F	15=O	24=X
7=G	16=P	25=Y
8=H	17=Q	26=Z

Then the message is converted to numbers and partitioned into **uncoded row matrices,** each having entries, as demonstrated in the Following Example

Example 2.27 Forming Uncoded Row Matrices

Write the uncoded row matrices of order for the message

MEET ME MONDAY.

Solution:

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{bmatrix}13 & 5 & 5\end{bmatrix}\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix}5 & 0 & 13\end{bmatrix}\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & 25 & 0\end{bmatrix}$$

M E E T M E M O N D A Y

Note that a blank space is used to fill out the last uncoded row matrix. To encode a message, use the techniques demonstrated in the following examples

"The Inverse of a Square Matrix" to choose an invertible matrix such as n × n"

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

and multiply the uncoded row matrices by (on the right) to obtain coded row matrices. Here is an example.

$$\begin{array}{ccc} \text{Uncoded Matrix} & \text{Encoding Matrix } A & \text{Coded} \\ \begin{bmatrix}13 & 5 & 5\end{bmatrix} & \begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix} & =\begin{bmatrix}13 & -26 & 21\end{bmatrix}\end{array}$$

Example 2.28 **Encoding a Message**

Use the following invertible matrix to encode the message MEET ME MONDAY.

$$A=\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}$$

Solution:

The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 2.28 by the matrix as follows.

$$\begin{array}{ccc} \text{Uncoded Matrix} & \text{Encoding Matrix } A^{-1} & \text{Coded Matrix} \\ \begin{bmatrix}13 & 5 & 5\end{bmatrix} & \begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix} & =\begin{bmatrix}13 & -26 & 21\end{bmatrix}\end{array}$$

$$\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}33 & -53 & -12\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}18 & -23 & 42\end{bmatrix}$$

$$\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}5 & -20 & 56\end{bmatrix}$$

$$\begin{bmatrix}1 & 25 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

So, the sequence of coded row matrices is

$$\begin{bmatrix}13 & -26 & 21\end{bmatrix}\begin{bmatrix}33 & -53 & -12\end{bmatrix}\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}5 & -20 & 56\end{bmatrix}$$
$$\begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

Finally, removing the matrix notation produces the following cryptogram

13 –26 21 33 –53 –12 18 –23 –42 5 –20 56 –24 23 77

Decoding a Message

For those who do not know the encoding matrix decoding the cryptogram found in Example 2.29 is difficult. But for an authorized receiver who knows the encoding matrix decoding is simple. The receiver just needs to multiply the coded row matrices by (on the right) to retrieve the uncoded row matrices.

Here is an example.

$$\begin{array}{ccc} \begin{bmatrix}13 & -26 & 21\end{bmatrix} & \begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix} & =\begin{bmatrix}13 & 5 & 5\end{bmatrix} \\ \text{Coded} & A^{-1} & \text{Uncoded}\end{array}$$

Example 2.29

Use the inverse of the matrix

$$A=\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix} \text{ to Decode the cryptogram}$$

13 – 26 21 33 –53 –12 18 –23 –42 5 – 20 56 –24 23 77

Solution:

$$\begin{array}{ccc} \text{Coded Matrix} & \text{Decoded Matrix } A^{-1} & \text{Decoded} \\ \begin{bmatrix}13 & -26 & 21\end{bmatrix} & \begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix} & =\begin{bmatrix}13 & 5 & 5\end{bmatrix}\end{array}$$

$$\begin{bmatrix}33 & -53 & -12\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}20 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}5 & -20 & -56\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}15 & 14 & 4\end{bmatrix}$$

$$\begin{bmatrix}-24 & -23 & -77\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}1 & 25 & 0\end{bmatrix}$$

First find A^{-1} by using the techniques demonstrated in the section entitled "The Inverse of a Square Matrix." A^{-1} is the decoding matrix. Then partition the message into groups of three to form the coded row matrices. Finally, multiply each coded row matrix by A^{-1} (on the right).

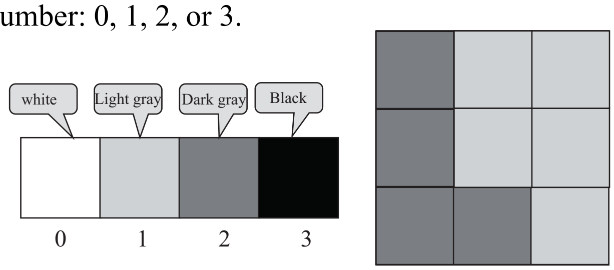
So, the message is as follows

$$\begin{bmatrix}13 & 5 & 5\end{bmatrix}\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix}5 & 0 & 13\end{bmatrix}\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & 25 & 0\end{bmatrix}$$

M E E T M E M O N D A Y

Example 2.30 Matrices and Digital Photography

The letter L in Figure is shown using 9 pixels in a 3 ×3 grid. The colors possible in the grid are shown in Figure . Each color is represented by a specific number: 0, 1, 2, or 3.



The Letter L

Figure 2.1

a. Find a matrix that represents a digital photograph of this letter L.

b. Increase the contrast of the letter L by changing the dark gray to black and the light gray to white. Use matrix addition to accomplish this

Solution:

a. Look at the L and the background in **Figure 2.1** Because the L is dark gray, color level 2, and the background is light gray, color level 1, a digital photograph of **Figure 2.1** can be represented by the matrix

$$\begin{bmatrix}2 & 1 & 1\\ 2 & 1 & 1\\ 2 & 2 & 1\end{bmatrix}$$

b. We can make the L black, color level 3, by increasing each 2 in the above matrix to 3. We can make the background white, color level 0, by decreasing each 1 in the above matrix to 0. This is accomplished using the following matrix addition:

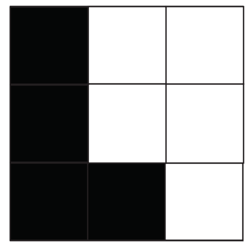

Changing contrast: the letter L

Figure 2.2

$$\begin{bmatrix}2 & 1 & 1\\ 2 & 1 & 1\\ 2 & 2 & 1\end{bmatrix}+\begin{bmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & 1 & -1\end{bmatrix}=\begin{bmatrix}3 & 0 & 0\\ 3 & 0 & 0\\ 3 & 3 & 0\end{bmatrix}$$ The picture corresponding to the matrix sum to the right of the equal sign is shown in **Figure 2.2**.

Example 2.31

The quadrilateral in **Figure 2.3** can be represented by the matrix. Each column in the matrix gives the coordinates of a vertex, or corner, of the quadrilateral. Use matrix operations to perform the following transformations:

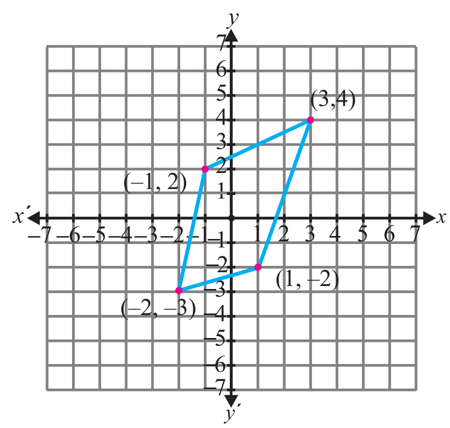


Figure 2.3

Coordinates of vertices

$$A=\begin{bmatrix}-2 & -1 & 3 & 1\\ -3 & 2 & 4 & -2\end{bmatrix}\begin{array}{l}\leftarrow \text{x-coordinates}\\ \leftarrow \text{y-coordinates}\end{array}$$

a. Move the quadrilateral 4 units to the right and 1 unit down.

b. Shrink the quadrilateral to half its perimeter.

c. Let $B=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. Find BA. What effect does this have on the quadrilateral in **Figure 2.3**?

Solution:

a. We translate the quadrilateral 4 units right and 1 unit down by adding 4 to each x-coordinate and subtracting 1 from each y-coordinate. This is accomplished using the following matrix addition:

$$\begin{bmatrix} -2 & -1 & 3 & 1 \\ -3 & 2 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 & 4 \\ -1 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 7 & 5 \\ -4 & 1 & 3 & -3 \end{bmatrix}$$

Each column in the matrix on the right gives the coordinates of a vertex of the translated quadrilateral. The original quadrilateral and the translated image are shown in **Figure 2.4**.

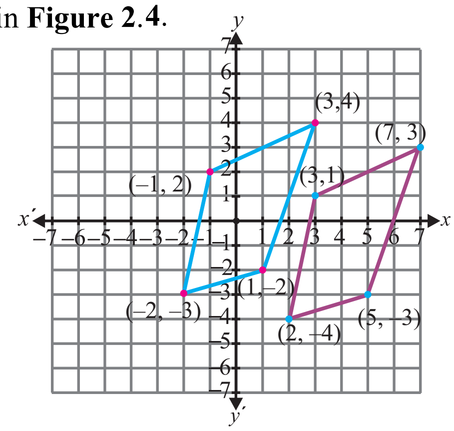


Figure 2.4

b. We shrink the quadrilateral in **Figure 2.3**, shown in blue in **Figure 2.5** to half its perimeter by multiplying each x-coordinate and each y-coordinate by $\frac{1}{2}$. This is accomplished using the following scalar multiplication:

$$\frac{1}{2}\begin{bmatrix} -2 & -1 & 3 & 1 \\ -3 & 2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & 1 & 2 & -1 \end{bmatrix}$$

Each column in the matrix on the right gives the coordinates of a vertex of the reduced quadrilateral The original quadrilateral and the reduced image are shown in **Figure 2.5**.

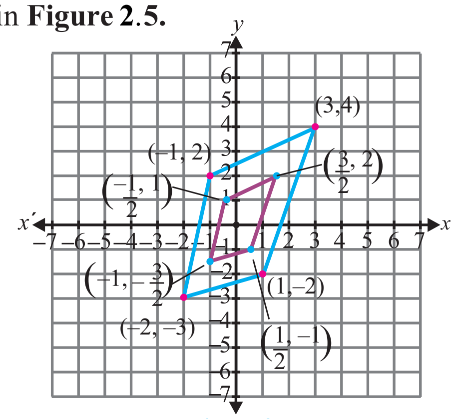


Figure 2.5

c. We begin by finding BA. Keep in mind that A represents the original quadrilateral, shown in blue in **Figure 2.6**

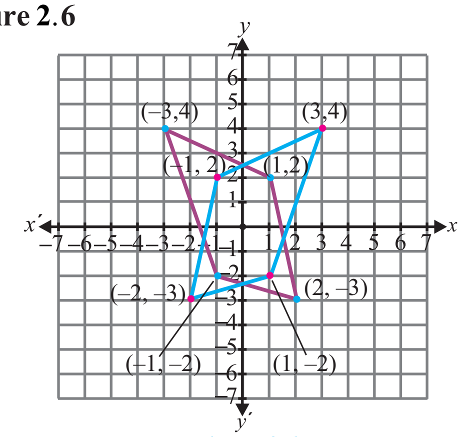


Figure 2.6

$$BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 & 3 & 1 \\ -3 & 2 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-2)+0(-3) & (-1)(-1)+0(2) & (-1)(3)+0(4) & (-1)(1)+0(-2) \\ 0(-2)+1(-3) & 0(-1)+1(2) & 0(3)+1(4) & 0(1)+1(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -3 & -1 \\ -3 & 2 & 4 & -2 \end{bmatrix}$$

Example 2.32

Table represents the number of active users (in thousands) on various social media platforms in four Pakistani cities:

City	Facebook	Twitter	Instagram
Karachi	150	80	120
Lahore	110	70	90
Islamabad	90	50	80

the average time spent on each platform in these cities is as follows (in minutes per day): Facebook: 40 minutes ; Twitter: 30 minutes

Instagram: 45 minutes. Calculate the total time spent on each social media platform in each city

Solution:

Multiplying the number of active users with the average time spent on each platform:

$$\begin{bmatrix} 40 & 30 & 45 \end{bmatrix}\begin{bmatrix} 150 & 80 & 120 \\ 110 & 70 & 90 \\ 90 & 50 & 80 \end{bmatrix}$$

$$= \begin{bmatrix} 6000+3300+4050 & 3200+2100+2250 & 5400+2700+3600 \end{bmatrix}$$

Total Active Facebook Users $= 6000+3300+4050$
$=13350$

Total Active Twitter Users $= 3200+2100+2250$
$= 7550$

Total Active Instagram Users $= 5400+2700+3600$
$= 11700$

Total Social Media Users $= \begin{bmatrix} 13350 & 7550 & 11700 \end{bmatrix}$

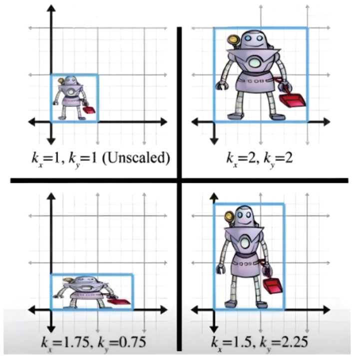
Total Minutes Spend in every City			
City	**Facebook**	**Twitter**	**Instagram**
Karachi	6000	3200	5400
Lahore	3300	2100	2700
Islamabad	4050	2250	3600

Remember that we are multiplying a 1×3 matrix with a 3×3 matrix and hence we get a 1×3 matrix.

Example 2.33 Transformation of Geometric Shapes

Problem Statement: Consider a 2D shape with vertices at points (1, 2), (3, 4), and (5, 6). Apply a scaling transformation to double the size of this shape.

Matrix in Computer Graphics



Solution:

1. **Represent the Vertices as a Matrix**:

Original Matrix: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

2. **Scaling Matrix**:

Scaling by 2: $S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

3. **Apply the Transformation**:

New Matrix: $A' = S \times A$

$$A' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}$$

Remember:		
Transformation	**Matrix Representation**	**Description**
Scaling	$\begin{bmatrix} Sx & 0 \\ 0 & Sy \end{bmatrix}$	Scales a shape by a factor Sx in the x-direction and Sy in the y-direction.
Reflection	About x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ About y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflects a shape across the x-axis or y-axis.

Conclusion: The new vertices of the scaled shape are (2, 4), (6, 8), and (10, 12).

Example 2.34 Data Encryption Using Matrices

Problem Statement: Encrypt the message "HI" using matrix multiplication. Let 'H' = 8 and 'I' = 9, and use the encryption matrix $E = \begin{bmatrix} 1 & 3 \\ -7 & 5 \end{bmatrix}$

Solution:

1. **Represent the Message as a Matrix**:

   Message Matrix: $M = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

2. **Apply the Encryption Matrix**:

   Encrypted Message: $M' = E \times M$

$$M' = \begin{bmatrix} 1 & 3 \\ -7 & 5 \end{bmatrix} \times \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

$$M' = \begin{bmatrix} 8+27 \\ -42+45 \end{bmatrix} = \begin{bmatrix} 35 \\ 3 \end{bmatrix}$$

Conclusion: The encrypted message is represented by the matrix $\begin{bmatrix} 35 \\ 3 \end{bmatrix}$

Skill 2.5

Real-world Application Proficiency:

- Apply matrix operations and solution methodologies to real-world problems involving linear transformations and modeling

Exercise 2.5

1. Scale a point (3, 5) in an image by a factor of 2 using a matrix.
2. Rotate a point (2, 2) in a graphic design by 90 degrees counterclockwise.
3. Given seismic data $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$ calculate the average reading for each sensor.
4. Reflect a point (5, 7) across the x-axis using a matrix.
5. Naveed and Sharjeel enroll in a Pakistani University A and the other university B, each with its own unique fee structure. They consider the number of credit hours they're taking and the cost per credit hour for the academic year 2022-2023 focusing on tuition expenses only.

	University A	University B
Naveed	6	9
Sharjeel	3	12

	Cost Per Credit Hour
University A	35000Rs
University B	42000 Rs

(a) Write a matrix A for the credit hours taken by each student and a matrix B for the cost per credit hour.

(b) Compute AB and interpret the results.

6. There are two shops in your area. Your shopping list consists of 2 kg of tomatoes, 5kg of meat, and 3 liters of milk. Prices differ between the different shops, and it is difficult to switch between shops to make certain you are paying the least amount of money. A better strategy is to check where you pay less on average. The prices of the different items are given in the table. Which shop should you go to?

Product	Price in shop A	Price in shop
Tomatoes	PKR 166/kg	PKR 158/kg
Meat	PKR 2550/Kg	PKR 2600/Kg
Milk	PKR 190/liter	PKR 220/liter

7. You are conducting a study on seismic waves in two regions. Your measurements consist of the speed (in km/s) of P-waves, S-waves, and surface waves. Differences exist in the seismic wave speeds between these regions, and it's crucial to determine which region exhibits overall faster

wave propagation for your study. The speeds of the seismic waves in the two regions are given in the table below:

$$A = \begin{array}{c} \begin{array}{cc} \text{Region A} & \text{Region B} \end{array} \\ \begin{bmatrix} 6.2\ \text{kms}^{-1} & 6\ \text{kms}^{-1} \\ 3.5\ \text{kms}^{-1} & 3.3\ \text{kms}^{-1} \\ 2.0\ \text{kms}^{-1} & 1.8\ \text{kms}^{-1} \end{bmatrix} \end{array} \begin{array}{l} \text{P-Wave} \\ \text{S-Wave} \\ \text{Surface Wave} \end{array} \Bigg|\ \text{Wave Type}$$

By analyzing the given seismic wave speeds in both regions, determine which region shows a higher average speed across P-waves, S-waves, and surface waves. This information will aid in understanding the comparative seismic wave propagation characteristics between Region A and Region B.

8. Cryptography: One method of encryption is to use a matrix to encrypt the message and then use the corresponding inverse matrix to decode the message. The encrypted matrix, E, is obtained by multiplying the message matrix, M, by a key matrix, K. The original message can be retrieved by multiplying the encrypted matrix by the inverse of the key matrix. That is $E = M \times K$ and $M = EK^{-1}$

(a) Given the key matrix $K = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 5 & 4 \end{bmatrix}$ find its inverse.

(b) Use your result from Part (a) to decode the encrypted matrix $E = \begin{bmatrix} 47 & 34 & 33 \\ 44 & 36 & 27 \\ 47 & 41 & 20 \end{bmatrix}$

(c) Each entry in your result for part (b) represents the position of a letter in the English Alphabet (A=1, B=2, C=3, and so on). What is the original message.

9. At a local dairy mart, the numbers of gallons of skim milk, 2% milk, and whole milk sold over the weekend are represented by

$$A = \begin{array}{c} \begin{array}{ccc} \text{Skim milk} & \text{2\% milk} & \text{Whole milk} \end{array} \\ \begin{bmatrix} 40 & 64 & 52 \\ 60 & 82 & 76 \\ 76 & 96 & 84 \end{bmatrix} \end{array} \begin{array}{l} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array}$$

The selling prices (in dollars per gallon) and the profits (in dollars per gallon) for the three types of milk sold by the dairy mart are represented by

$$B = \begin{array}{c} \begin{array}{cc} \text{Selling Price} & \text{Profit} \end{array} \\ \begin{bmatrix} 40 & 64 \\ 60 & 82 \\ 76 & 96 \end{bmatrix} \end{array} \begin{array}{l} \text{Skim milk} \\ \text{2 \% milk} \\ \text{Whole milk} \end{array}$$

(a) Compute AB and interpret the result.

(b) Find the dairy mart's total profit from milk sales for the weekend.

10. A company that manufactures boats has the following labor-hour and wage requirements.

$$P = \begin{array}{c} \text{Department} \\ \begin{array}{ccc} \text{Cutting} & \text{Assembly} & \text{Packaging} \end{array} \\ \begin{bmatrix} 1\ hr & 0.5\ hr & 0.2\ hr \\ 1.6\ hr & 1\ hr & 0.2\ hr \\ 2.5\ hr & 2\ hr & 1.4\ hr \end{bmatrix} \end{array} \begin{array}{l} \text{Small} \\ \text{Medium} \\ \text{Large} \end{array} \Bigg|\ \text{Boat size}$$

Wages Per hour

$$Q = \begin{array}{c} \begin{array}{cc} \text{Plant A} & \text{Plant B} \end{array} \\ \begin{bmatrix} 20\ Rs & 16\ Rs \\ 16\ Rs & 12\ Rs \\ 12\ Rs & 8\ Rs \end{bmatrix} \end{array} \begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array} \Bigg|\ \text{Department}$$

Compute PQ and interpret result

11. Jamal and Saleem each have student loans issued from the same two banks. The amounts borrowed and the monthly interest rates are given next (interest is compounded monthly):

	Lender 1	Lender 2
Jamal	350000	300000
Saleem	450000	380000

	Monthly Interest Rate
Lender 1	0.011 (1.1%)
Lender 2	0.006 (0.6%)

(a) Write a matrix A for the amounts borrowed by each student and a matrix B for the monthly interest rates.

(b) Compute AB and interpret the results.

(c) Let $C = \begin{bmatrix}1\\1\end{bmatrix}$ Compute A(C + B) and interpret the results.

12. A company sell five models of computers through three retail outlets. The inventories are represented by A & the Wholesale and retail prices are represented by B. Computer AB.

Model

$$A = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 3 & 2 & 3 & 0 & 4 \\ 0 & 2 & 4 & 3 & 2 \\ 4 & 2 & 4 & 2 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Price

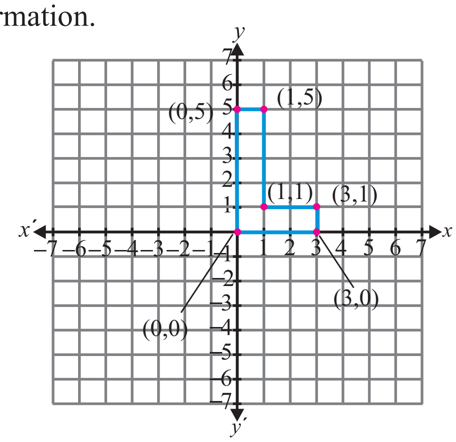
$$B = \begin{array}{c} \begin{array}{cc} \text{Wholesale} & \text{Retail} \end{array} \\ \begin{bmatrix} 840 & 1100 \\ 1200 & 1350 \\ 1450 & 1650 \\ 2650 & 3000 \\ 3050 & 3200 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ .C \\ D \\ E \end{array}$$

13. The figure shows the letter L in a rectangular coordinate system. The figure can be represented by the matrix $B = \begin{bmatrix} 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 5 \end{bmatrix}$

Each column in the matrix describes a point on the letter. The order of the columns shows the direction in which a pencil must move to draw the letter. The L is completed by connecting the last point in the matrix, (0, 5), to the starting point, (0, 0). Use these ideas to solve following

(a) Use matrix operations to move the "L" 2 units to the right and "3" units down. Then graph the letter and its transformation in a rectangular coordinate system.

(b) Reduce the L to half its perimeter and move the reduced image 1 unit up. Then graph the letter and its transformation.



Review Exercise 2

Each of the questions or incomplete statement below is followed by four suggested answers or completions. In each case, select the one that is the best of the choices.

(i) The order of matrix $A = \begin{bmatrix} a & b \\ d & e \\ h & i \end{bmatrix}$ is:

(a) 2×3 (b) 6×3 (c) 3×2 (d) 3×6

(ii) A matrix having all elements are zero is known as____________

(a) scalar matrix (b) diagonal matrix (c) square matrix (d) null matrix

(iii) A matrix having rows and columns are equal then it is called:

(a) row matrix (b) column matrix (c) square matrix (d) rectangle matrix

(iv) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is:

(a) a row matrix (b) an identity matrix (c) scalar matrix (d) diagonal matrix

(v) A matrix whose at least one element is imaginary number then this matrix is called:

(a) real matrix (b) natural matrix (c) complex matrix (d) none of the given option

(vi) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ then A^t is

(a) $\begin{bmatrix} a_{13} & a_{21} \\ a_{12} & a_{22} \\ a_{11} & a_{23} \end{bmatrix}$ (b) $\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$ (c) $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$ (d) $\begin{bmatrix} a_{21} & a_{11} \\ a_{22} & a_{12} \\ a_{23} & a_{13} \end{bmatrix}$

(vii) A square matrix X is said to be skew symmetric if:

(a) $X^t = X$ (b) $X^t = t\sqrt{X}$ (c) $X^t = -X$ (d) $X^t = \dfrac{t}{X}$

(viii) A square matrix X is said to be a symmetric if:

(a) $X^t = X$ (b) $X^t = t\sqrt{X}$ (c) $X^t = -X$ (d) $X^t = \dfrac{t}{X}$

(ix) If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and k is any scalar then $kA =$ ___________

(a) $\begin{bmatrix} kp & q \\ r & s \end{bmatrix}$ (b) $\begin{bmatrix} kp & q \\ r & ks \end{bmatrix}$ (c) $\begin{bmatrix} kp & kq \\ kr & ks \end{bmatrix}$ (d) $\begin{bmatrix} kp & kq \\ r & ks \end{bmatrix}$

(x) The product of A and B is _________ if $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(a) $\begin{bmatrix} -4 & -4 \\ 32 & 77 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 32 \\ 32 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & 32 \\ 5 & 32 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 32 \\ 32 & -4 \end{bmatrix}$

(xi) If A is a matrix of order 2×3 and B is a matrix of 3×4 then the order of AB is:

(a) 3×3 (b) 3×4 (c) 2×3 (d) 2×4

(xii) If A is a matrix of order 3×2 then the order of its transpose matrix will be:

(a) 3×2 (b) 3×3 (c) 2×3 (d) 2×2

(xiii) Let $A = [5 \quad 6]$ and $B = [0 \quad -1]$ then $3A + 2B$ is:

(a) $[16 \quad 15]$ (b) $\begin{bmatrix} 16 \\ 15 \end{bmatrix}$ (c) $[15 \quad 16]$ (d) $\begin{bmatrix} 15 \\ 16 \end{bmatrix}$

(xiv) Which property does not holds in matrix:

(a) commutative property w.r.t addition

(b) associative property w.r.t addition

(c) commutative property w.r.t multiplication

(d) additive inverse

(xv) A given matrix X is singular if:

(a) $|X| = 0$ (b) $|X| \neq 0$ (c) $|X| < 0$ (d) $|X| > 0$

(xvi) A given matrix X is non-singular if:

(a) $|X| = 0$ (b) $|X| \neq 0$ (c) $|X| < 0$ (d) $|X| > 0$

(xvii) $AA^{-1} =$ ________

(a) A^2 (b) A (c) A^{-1} d) I

(xviii) The order of the matrix [365] is:

(a) 3×3 (b) 3×1 (c) 1×3 (d) 1×1

(xix) If $A = \begin{bmatrix} 4 & 5 \\ 6 & 7 \\ 8 & 1 \end{bmatrix}$ then A^{-1} is:

(a) $\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -6 \\ 8 & -5 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 5 \\ \frac{1}{8} & 3 \\ 9 & 2 \end{bmatrix}$ (d) Not possible

(xx) Two matrices X and Y are multiplied to get XY if:

(a) both matrix are rectangular

(b) both matrix are square matrix

(c) both matrix have same order

(d) no. of columns of matrix X are equal to the no. of rows of matrix Y

2. Perform the Following Operations

(i) $2 \times \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ -2 & 1 & 4 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 4 & 1 & 3 \\ -2 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ 0 & 5 & -1 \\ 2 & 3 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

3(a). Perform row operations to reduce the matrix M to its echelon form $\mathbf{M} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-3} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{0} \end{bmatrix}$

(b). What are the elementary row operations applied to M to reach its echelon form

4(a). Reduce the matrix N to its reduced echelon form $\mathbf{N} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{7} \\ \mathbf{2} & \mathbf{1} & \mathbf{10} \\ \mathbf{3} & \mathbf{0} & \mathbf{15} \end{bmatrix}$

(b) Describe the Sequence of row operations used to achieve the reduced Echelon for of N.

5. Identify whether the following systems of equations are homogeneous or non-homogeneous:

(a):

$2x-y+3z=0$; $3x+2y-z=5$; $x+z=2$

(b):

$3x-2y+z=4$; $4x+y+2z=10$; $2x+z=-1$

6. **Determine if the following systems of equations are consistent or inconsistent:**

(a):

$2x+y-z=3$; $4x+2y+z=6$; $x-2y=1$

(b):

$3x-2y+z=4$; $6x-4y+2z=7$; $2x+y-z=0$

7. **Solve the following system of equations using the matrix inversion method:**

$2x + 3y - z = 1$; $4x - y + 2z = 6$; $-x + 2y - z = 4$

(a) Find the solution for x, y, and z using matrix inversion.

(b) Check the consistency of the system.

8. **Apply the Gauss elimination method to solve the system:**

$3x + 2y + z = 5$; $2x - 3y + z = -1$; $4x + y - 2z = 9$

(a): Determine the solution for x, y, and z using Gauss elimination.

(b): Is the system consistent or inconsistent?

9. Apply matrix operations and solution methods to solve the following real-world problem. system of equations representing the flow of goods in a supply chain:

$2x + 3y - z = 50$; $30x + 2y + z = 30$; $3x - y + 2z = 70$

(a): Determine the quantities of products (x, y, and z) involved in the supply chain.

(b): Analyze the consistency of the supply chain flow based on the solution.