A Textbook of

Mathematics

10th Grade

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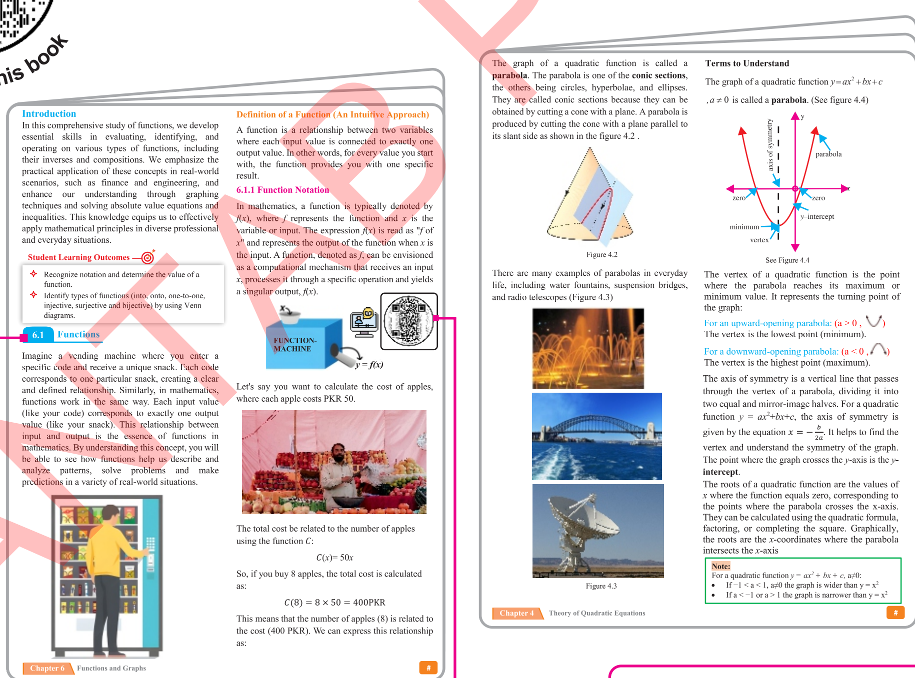
Preface

In this dynamic 10th-grade mathematics textbook, I embrace the evolving world of education by utilizing the CPA (Concrete, Pictorial, Abstract) Approach. This method, grounded in concrete examples, pictorial representations, and abstract concepts, caters to diverse learning styles, making mathematics accessible and engaging. Interactive images and real-life examples transform mathematical theories into vivid, relatable experiences, enhancing understanding and enjoyment.

The book encourages active learning through "Test Yourself" sections, classroom activities promoting collaboration and critical thinking, and insightful "Teacher's Footnotes" for effective content delivery. Rich in interactive color images, it offers a visually stimulating learning environment, breaking the monotony of traditional texts.

With a variety of examples, worksheets, and video lectures, the textbook provides comprehensive practice and learning opportunities. Additionally, simulations allow hands-on exploration of concepts, deepening understanding. This textbook is more than an educational tool; it's a journey designed to instigate a deep appreciation for mathematics, connecting the subject with the rhythm of the modern educational landscape.

How to use this book



The purpose of a skill is to apply knowledge. Students and teachers can scan the provided QR code to access a worksheet that enhances their understanding.

Knowledge is information about a specific topic that helps clarify concepts. Students and teachers can scan the QR code provided with the SLO Tab to access lectures related to that topic.

The CPA approach, enhanced with interactive images, makes mathematics more accessible and engaging, transforming abstract concepts into tangible visuals for deeper understanding and effective learning.

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SLO based Model Video lecture

Salient Features

Comprehensive Learning
Engage students with videos, simulations, and practical worksheets.

Structured Lesson Plan
Well-organized with clear objectives, PPTs, and a question bank.

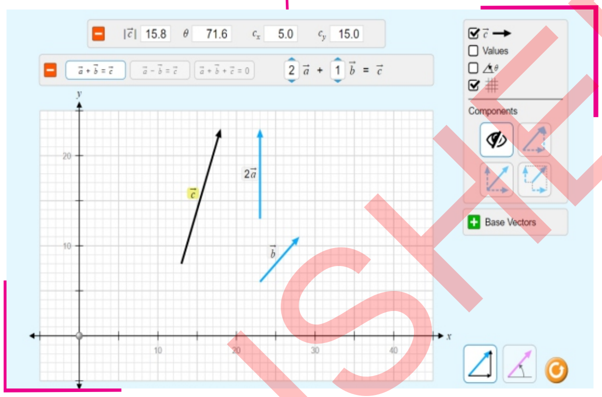
Engaging Multimedia
Visual appeal through PPTs and interactive simulations.

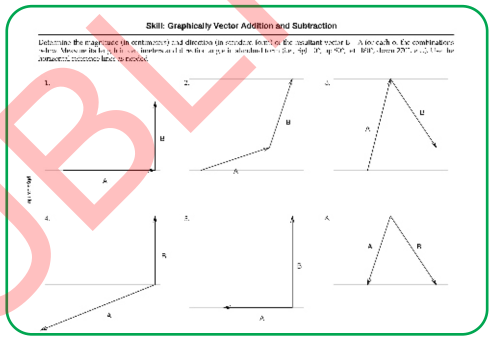
Assessment & Tracking
Diverse question bank and progress monitoring.

Adaptable & Accessible
Scalable and accessible, suitable for all learners.

SLO No: M - 10 - B - 07
SLO statement: Addition and Subtraction of vectors

Simulation





SKILL SHEET

SKILL

Skill:8.1

- Understanding the rectangular coordinate system for 2D planes.
- Representing vectors as directed line segments with magnitude and direction.
- Utilizing position vectors to describe point locations.
- Calculating vector magnitudes in specific directions.

Exercise 8.1

1. Classify the following measurement as scalars and vectors.

(i) 5 kg (ii) 3 meters East North
(iii) 25-watts (iv) 35°C
(v) 40 m/sec^2

2. Find the magnitude of the following vectors.

(i) $\overline{P}(2,3)$ (ii) $\overline{Q}(4,7)$
(iii) $\overline{R}(-2,9)$ (iv) $\overline{S}(-2,-2)$
(v) $\overline{T}(0,-7)$

Student Learning Outcomes

- Add and subtract vectors.
- Multiply a vector by a scalar.
- Express translation by a vector.
- Express a vector in terms of two non-zero and non-parallel coplanar vectors.

8.10 Addition and Subtraction of Vectors

KNOWLEDGE

8.10.1: Addition of Vectors

Two given vectors can be added by the following three laws.

Head – to – Tail or Triangle Law of Addition

To add two vectors, $\overline{x}$ and $\overline{y}$, we position them such that the head of the first vector coincides with the tail of the second vector. The resultant vector, $\overline{x}+\overline{y}$, is obtained by drawing a vector from the tail of the first vector to the head of the second vector, as shown in figure 8.12. This method of adding vectors is known as the Head-to-Tail method or the Triangle Law of Vector Addition.

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CHAPTER 1 Complex Numbers

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Did you know that complex numbers play a vital role in signal processing, a key aspect of contemporary communication technology? These numbers are crucial for understanding and manipulating signals in various forms of data transmission. Complex numbers are used to accurately represent the phase and amplitude of signal components, enabling engineers to efficiently filter, compress and reconstruct signals. This application is critical in a range of technologies, from improving the clarity of mobile phone communications to ensuring the accuracy of satellite transmissions. The use of complex numbers in signal processing highlights their invaluable contribution to advancing communication technology.

Students' Learning Outcome

1 Identify complex numbers, complex conjugate, absolute value or modulus of a complex number
2 Apply algebraic properties and perform basic operations on complex numbers
3 Demonstrate additive identity and multiplicative identity for the set of complex numbers
4 Find additive inverse and multiplicative inverse of a complex number z.
5 Demonstrate the following properties of a complex number

$$|z| = |-z| = |\overline{z}| = |-\overline{z}|,$$

$$\overline{\overline{z}} = z,\ \overline{z}\,\overline{z} = |z|^2,\ \overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2,$$

$$\overline{z_1 z_2} = \overline{z}_1 + \overline{z}_2,\quad \overline{\left(\frac{z_1}{z_2}\right)} + \frac{\overline{z}_1}{\overline{z}_2},\ z_2 \neq 0.$$

6 Find real and imaginary parts of the following types of complex numbers:
$(x + iy)^n$, $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n, x_2 + iy_2 \neq 0$, where n = 1 and 2

7 Solve the simultaneous linear equations with complex coefficients,
8 Apply the Geometric interpretation of a complex number
9 Apply the geometric interpretation of the modulus of a complex number.
10 Apply the geometric interpretation of algebraic operations.

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Knowledge

- **Understanding Complex Numbers:** Comprehending that a complex number is in the form $z = a + bi$ where a and b are real numbers and i is the imaginary unit.
- **Complex Conjugates:** Knowing how to find the complex conjugate of a given complex number, $z = a + b\bar{i}$ which is $z = a - bi$.
- **Modulus of Complex Numbers:** Understanding the modulus (or absolute value) of a complex number as $|\ z\ | = \sqrt{a^2 + b^2}$, representing its distance from the origin in the complex plane
- **Properties of Complex Numbers:** Familiarity with algebraic properties such as distributive, associative,and commutative properties in the context of complex numbers.
- **Demonstrate properties of Conjugate of a complex number z**

$$|z|=|-z|=|\bar{z}|=|-\bar{z}|,\ \bar{\bar{z}} = z,\ z\bar{z} = |z|^2,\ \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 + \bar{z}_2,\quad \overline{\left(\frac{z_1}{z_2}\right)} + \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0.$$

- **Real and Imaginary Parts of Powers of Complex Numbers:** Knowledge of how to expand and interpret expressions like $(x + iy)^n$ for different values of n to get real and imaginary parts
- **Solving Complex Linear Equations:** Skills in solving linear equations with complex coefficients using methods like substitution, elimination, or matrix approaches.
- **Geometric Interpretation:** Understanding the representation of complex numbers in the Argand diagram and visualizing their algebraic operations geometrically.

Pre & Post Requisite

9^{th} Class Math
Chapter # 1
Real Number System

10^{th} Class Math
Chapter # 1
Complex Numbers

1^{st} year Class Math
Chapter # 1
Complex Numbers

Skill

- **Understanding Complex Numbers:** Proficiency in comprehending that a complex number is represented in the form $z=a+bi$, where a and b are real numbers, and i is the imaginary unit.
- **Finding Complex Conjugates:** Ability to calculate the complex conjugate of a given complex number, $z=a+bi$, resulting in $\bar{z}=a-bi$.
- **Calculating Modulus of Complex Numbers:** Skill in determining the modulus (or absolute value) of a complex number as $|\ z\ | = \sqrt{a^2 + b^2}$, signifying its distance from the origin on the complex plane.
- **Understanding Properties of Complex Numbers:** Proficiency in applying algebraic properties, including distributive, associative, and commutative properties,within the context of complex numbers.
- **Demonstrating Properties of Conjugate:** Ability to illustrate the properties and applications of the conjugate of a complex number z.
- **Analyzing Real and Imaginary Parts of Complex Powers:** Skill in expanding and interpreting expressions like $(x+iy)^n$ for various values of n to extract the real and imaginary parts.
- **Solving Complex Linear Equations:** Proficiency in solving linear equations with complex coefficients using methods such as substitution, elimination, or matrix approaches.
- **Utilizing Geometric Interpretation:** Application of geometric interpretation to understand the representation of complex numbers in the Argand diagram and to visualize algebraic operations geometrically.

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Student Learning Outcomes

- Identify complex numbers, complex conjugate, absolute value or modulus of A complex number

In this chapter, we'll uncover the world of complex numbers, learning how to find their conjugates, moduli, and explore their algebraic properties. You'll master solving complex linear equations and visualize these numbers on the Argand diagram. By the end, you'll see how complex numbers elegantly solve real-world problems and enhance mathematical understanding.

1.1 Complex Numbers

A fundamental property of real numbers is that their squares are always nonnegative. For instance, there is no real number x such that $x^2 = -1$. To address this, mathematicians introduced the imaginary unit, denoted by i, which is defined by the property:

$$i^2 = -1$$

This introduction of i continues the historical pattern of expanding number systems to solve problems that couldn't be addressed within existing systems. For example, if we only had integers, equations like $3x = 1$ would have no solution. This limitation led to the creation of rational numbers, such as $\frac{1}{3}$ and $\frac{2}{5}$. Similarly, within a universe of rational numbers, equations like $x^2 = 3$ have no solutions, leading to the introduction of irrational numbers, such as $\sqrt{3}$ and $\sqrt{7}$. The real numbers, therefore, include both rational and irrational numbers. Now, within the universe of real numbers, there is no solution to $x^2 = -1$, prompting the introduction of the imaginary unit i, where $i^2 = -1$.

Through this exploration, we see that each new number system-from integers to rational numbers, to real numbers, and finally to complex numbers-has been developed to solve equations that could not be addressed by previous systems.

A simple consequence of the definition of i is that all powers of i may be expressed in terms of ± 1 or $\pm i$ itself.

For example, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2) = 1$ and if we continue in this way to obtain higher powers of i, we obtain the values $1, i, -1$ or $-i$. Similarly, for negative powers, we have

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i \qquad \therefore i^2 = -1$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1 \qquad i^{-3} = \frac{1}{i^3} = \frac{1}{i^2 i} = \frac{1}{-i} = \frac{1}{-i.i} = i$$

and so on.

i^{-4}	i^{-3}	i^{-2}	i^{-1}	i^0	i^1	i^2	i^3
1	i	-1	$-i$	1	i	-1	$-i$

1.1.1 Complex Numbers represented by an expression of the form $z = a + ib$

The complex number system arises from incorporating i. Complex numbers are expressed in the form $a + bi$, where a and b are real numbers. In this expression, a is called the **real part**, b is the **imaginary part**, and i is the imaginary unit, with $i^2 = -1$.

For example, in the complex number $2 + 4i$, 2 is the real part, and 4 is the imaginary part. When a complex number has a negative imaginary part, such as $5 + (-3)i$, it is typically written as $5 - 3i$. Additionally, the complex number $a + 0i$ is usually written simply as a, indicating that real numbers are a subset of complex numbers as we can see in the figure 1.1. Similarly, $0 + bi$ is written as bi, and such numbers are often referred to as pure imaginary numbers.

Usually, the complex number $x + yi$ is denote by $z = x + yi$

Accordingly, $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$, $z_3 = x_3 + y_3 i$, …

The set of all complex numbers is denoted by $\mathbb{C}$, that is

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$$

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Note:
In the expression $x + yi$:
Real Numbers: If $y = 0$, then $x + yi = x$. Every real number x can be written as a complex number with $y = 0$.
Pure Imaginary Numbers: If $x = 0$ and $y \neq 0$, then $x + yi = yi$, y known as a pure imaginary number (e.g., i and $-i$).
Zero Complex Number: When both x and y are zero, $x + yi = 0$.
Unit Complex Number: When $x = 1$ and $y = 0, x + yi = 1$.
Thus, all real numbers can be expressed as complex numbers in the form $x + 0i$.

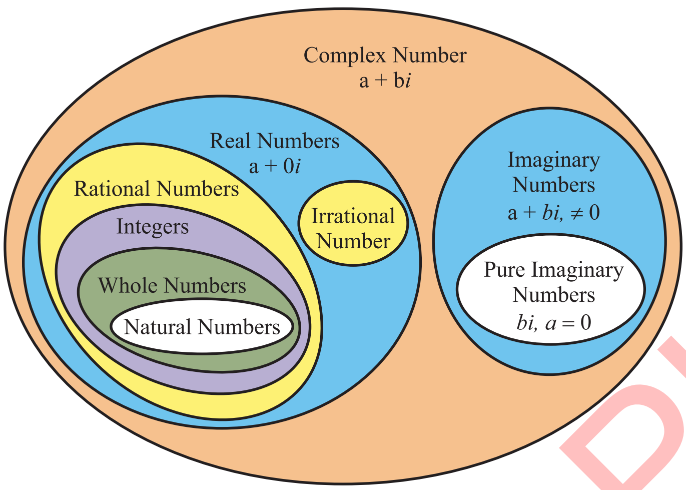

Figure 1.1: Depicting the set of complex numbers and related subsets

1.1.2 Complex Numbers as Ordered Pairs of Real Numbers

Complex numbers may also be defined as ordered pairs of real numbers. Thus a **complex number** z is an ordered pair (a, b) of real numbers a and b, written as $z = (a, b)$

The first component a is called the **real part** of z and the second component b is called the **imaginary part.** The real part is denoted by Re(z) and imaginary part is denoted by Im(z) respectively i.e. Re(z) = a and Im(z) = b. The ordered pair (0,1) is known as **imaginary unit** and it is denoted by $\boldsymbol{i} = \mathbf{(0,1)}$

The set of all ordered pairs of real numbers is the set of complex numbers denoted by C, that is

$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\} = I\!R \times I\!R$, where $I\!R$ is the set of real numbers.

We are in a position to express every complex number z as an ordered pair in terms of i as follows;

$$z = (a, b) = (a, 0) + (0, b)$$
$$= (a, 0) + (b, 0)(0,1)$$
$$= a + bi \qquad (\because a = (a, 0) \text{ and } i = (0, 1)$$

that is $\quad z = (a,b) = a + bi$

We see that an ordered pair (a,b) is expressible in the usual form of complex number as $a + bi$. Thus the two notations for a complex numbers z can be used interchangeably.

Example 1.1: Write in the form of $a+bi$ and (a,b)

(i) 7 **(ii)** $3i$ **(iii)** 0

(iv) $\frac{1}{2}$ **(v)** $3 - \sqrt{-16}$ **(vi)** 1.

Solution:

(i) $7 = 7 + 0i = (7,0)$

(ii) $3i = 0 + 3i = (0,3)$

(iii) $0 = 0 + 0i = (0,0)$

(iv) $\frac{1}{2} = \frac{1}{2} + 0i = \left(\frac{1}{2}, 0\right)$

(v) $3 - \sqrt{-16} = 3 - i\sqrt{16} = 3 - 4i = (3, -4)$

(vi) $1 = 1 + 0i = (1, 0)$

Student Learning Outcomes

- Apply the Geometric interpretation of a complex number
- Apply the geometric interpretation of the modulus of a complex number.

1.2 Graphical Representation of Complex Numbers

The complex plane is also known as "The Argand Diagram" after the French-Swiss Mathematician Jean Robert Argand (1768-1822).
An Argand diagram is a graphical representation of complex numbers on a two-dimensional plane.

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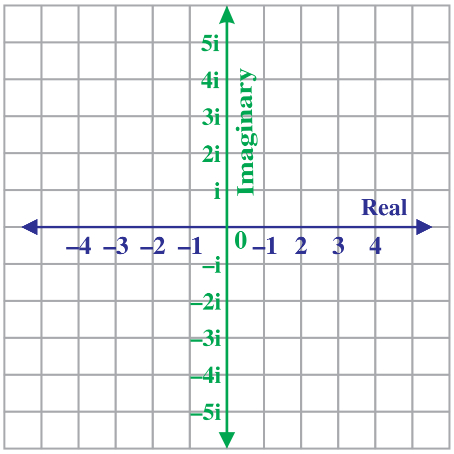


Figure 1.2

The horizontal axis (commonly referred to as the **real axis**) represents the real part of a complex number (see figure 1.2).
The vertical axis (known as the **imaginary axis**) represents the imaginary part of the complex number. A complex number $z = a + bi$ **(where a and b are real numbers, and i is the imaginary unit)** is represented as a point on this plane, with a determining the position along the real axis, and b determining the position along the imaginary axis as shown in figure 1.3.

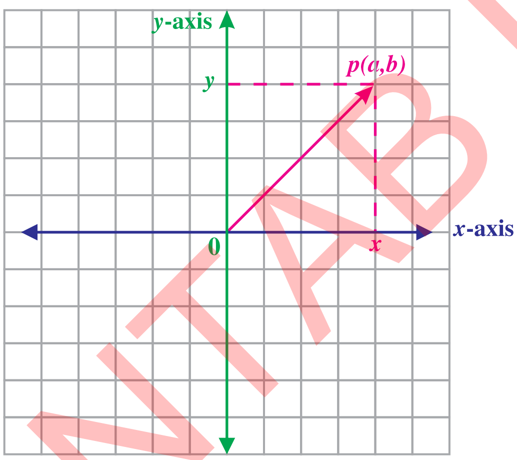


Figure 1.3

Complex numbers offer a captivating geometric perspective on standard arithmetic operations. While real numbers are represented on a one-dimensional number line, providing certain insights into their characteristics, the advent of the imaginary unit i allows us to extend this into a two-dimensional plane. This approach of visualizing complex numbers as points on a plane opens up new avenues for understanding their unique properties.

Every point in the plane may be associated with just one complex number. Thus, there is one-one correspondence between the infinite set of complex numbers and the points of the plane.

Example 1.2: Represent the following complex numbers in the complex plane $-4+3i$, $3+2i$, $-2-3i$ and $5-5i$.

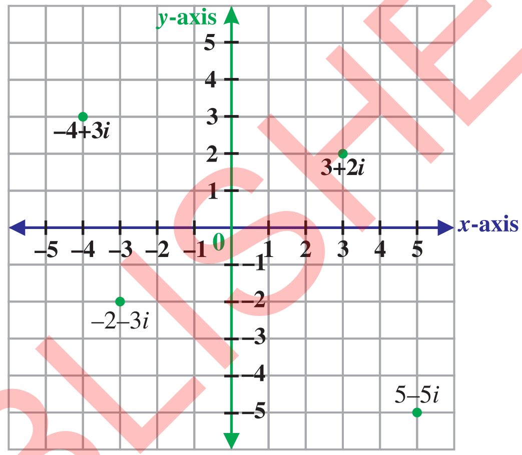


Figure 1.4

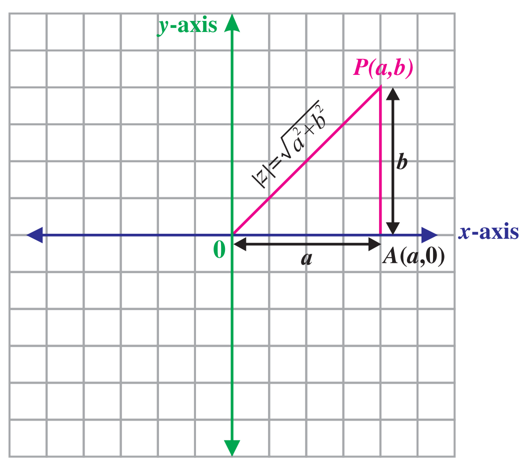
Solution: In figure 1.4, all the above complex numbers have been represented. We see that the complex numbers appear in all the four quadrants due to the negative and positive signs with their real and imaginary parts.

1.1.2 Absolute value or Modulus of a Complex Number

Let $z = (a,b) = a+bi$ be a complex number. Then **absolute value** or **modulus** of z, denoted by $|z|$, is defined by

$$|z| = \sqrt{a^2 + b^2}$$

In the adjoining figure P represents $a + bi$. $\overline{PQ}$ is a perpendicular drawn on x-axis.



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Thus $\overline{OQ} = a$ and $\overline{PQ} = b$. In the right angled–triangle OQP, we have, by Pythagoras theorem

$$|\overline{OP}|^2 = |\overline{OQ}|^2 + |\overline{PQ}|^2$$

$$|\overline{OP}| = |z| = \sqrt{a^2 + b^2}$$

Therefore, the modulus of a complex number is the distance from the origin of the point representing the number.

Note: The modulus of a complex number is always positive because it represents the distance from the origin in the complex plane, which is always non-negative. It is zero only when the complex number itself is zero.

Example 1.3: Compute the absolute value of the following complex numbers:

(i) $2i$ **(ii)** 4 **(iii)** $3 - 6i$

Solution:

(i) Let $z = 2i$ or $z = 0 + 2i$

Then by the definition

$$|z| = \sqrt{(0)^2 + (2)^2} = \sqrt{2^2} = 2$$

(ii) Let $z = 4$ or $z = 4 + 0i$.

Then by the definition

$$|z| = \sqrt{(4)^2 + (0)^2} = \sqrt{4^2} = 4$$

(iii) Let $z = 3 - 6i$.

Then by the definition

$$|z| = \sqrt{(3)^2 + (-6)^2} = \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$$

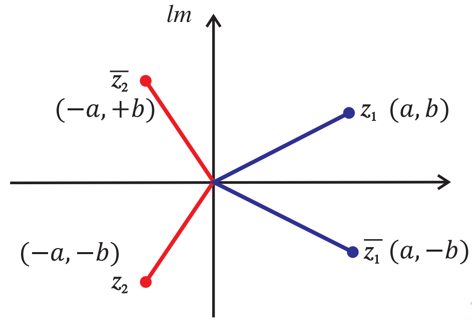
1.3 Conjugate of a Complex Number

The conjugate of the complex number $a + bi$ is $a - bi$ and $a - bi$ is $a + bi$.

We denote the conjugate of any complex number z as $\overline{z}$. is obtained by changing the sign of the imaginary part of z.

Thus Conjugate of $z = a + bi$ is $\overline{z} = a - bi$

Geometrically speaking, the conjugate of z is simply a reflection of z about the real axis, as shown below:



Example 1.4: Find the conjugate of

(i) $-4 - 5i$ and **(ii)** $6 + 9i$.

Solution:

(i) $\overline{-4 - 5i} = -4 + 5i$ **(ii)** $\overline{6 + 9i} = 6 - 9i$

Student Learning Outcomes

- Apply algebraic properties and perform basic operations on complex numbers
- Apply the geometric interpretation of algebraic operations.

1.4 Basic Algebraic Operations on Complex Numbers.

In this section, we explore the core arithmetic operations: addition, subtraction, multiplication, and division, as they apply to understanding how these complex numbers. This study is essential for numbers, which combine real and imaginary components, function within the realm of advanced mathematics

Let $z_1 = a_1 + b_1 i$ and $z_2 = c + b_2 i$ be two complex numbers. Then their,

(i) Addition:

Adding complex numbers involves summing their real and imaginary parts separately.

For $z_1 = a + bi$ and $z_2 = c + di$ the sum is $(a + c) + (b + d)i$

Geometrically, on the Argand Diagram, each complex number is represented as a point or vector. Adding them involves placing the second vector's tail at the head of the first vector. The resultant vector from the origin represents the sum. For example, $z_1 = 3 + 4i$ and $z_2 = 1 + 2i$ sum to $4 + 6i$ when plotted and added geometrically, providing a clear visual representation of the operation.

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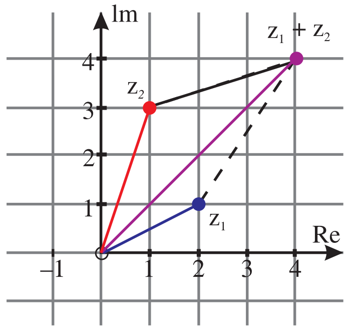
$$z_1 + z_2 = (a+bi)+(c+di)=(a+c)+(b+d)i$$

Geometrical Representation

Let us now study the geometric effect of adding/subtracting one complex to/from another complex number. For example, we can plot 2 + i and 1 + 3 i on an Argand diagram. Plotting the sum (2 + i) + (1 + 3 i)= 3 + 4i shows us that addition of two complex numbers is the same as addition of vector, and can be done via the parallelogram law:



This also show us that addition of complex numbers is commutative, i.e. $z_1 + z_2 = z_2 + z_1$. It is important to know this since we cannot assume that anything about the arithmetic of real numbers transfers to the arithmetic of complex numbers.

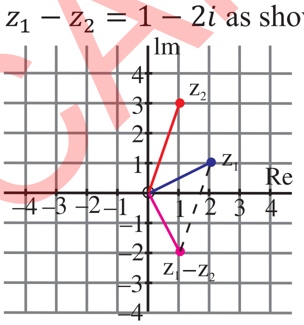
(ii) Subtraction:

Just Like Addition of complex numbers Real part subtracted from real while imaginary subtracted from imaginary. Consider two imaginary numbers $z_1 = a + bi$ and $z_2 = c + di$ there difference will be

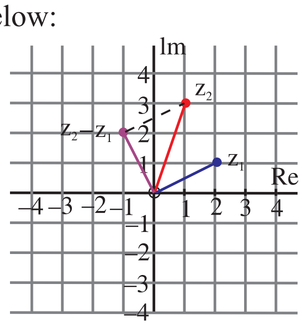
$$z_1 - z_2 = (a+bi)-(c+di)=(a-c)+(b-d)i$$

Geometrical Representation

For Graphical interpretation parallelogram law applies for subtraction as well. For example, if $z_1 = 2 + i$ and $z_2 = 1 + 3i$ then $z_2 - z_1 = -1 + 2i$ and $z_1 - z_2 = 1 - 2i$ as shown below:



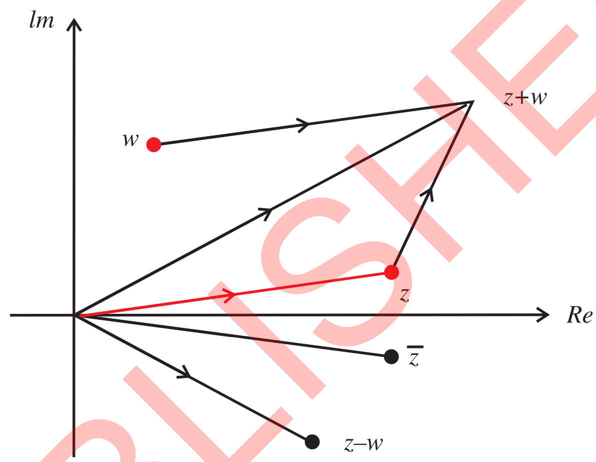
$z_1 - z_2 = 1 - 2i$



$z_2 - z_1 = -1 + 2i$

Hence proved geometrically and algebraically, that subtraction is not commutative.

Below is an all-in-one diagram representing the geometric effect of the three operations of conjugation, addition, and subtraction on two complex numbers z and w.



Example 1.5: Perform the indicated operation in each of the following.

(i) (8 – 5i) + (5 + 6i) **(ii)** i – (6 – 9i).

Solution:

(i)(8 – 5i) + (5 + 6i) = (8 + 5) + (–5 + 6)i = 13 + i

(ii)i – (6 – 9i) = (0 – 6) + (1 – (–9))i = – 6 + 10i

(iii) Multiplication:

1.7.1 Multiplying Two Complex Numbers

When we think about multiplying real numbers, it's a simple operation-multiply the values together, like 5×10=50. But with complex numbers, which have both a real part and an imaginary part, the process is a bit different.

Multiplication of a scalar with a complex number

In mathematics, when we multiply a real number (also known as a scalar) by a complex number, we are essentially scaling the complex number. The

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operation is straightforward: the scalar multiplies both the real and imaginary parts of the complex number.

For example

$kz_1 = k(a_1 + b_1 i) = ka_1 + kb_1 i$ for any real number k.

Now,let us understand how do we multiply numbers like $a+ib$ and $c+id$? Fir this consider two complex numbers $z_1=a+ib$ and $z_2=c+id$. We define the product z_1z_2 to be:

$$z_1z_2=(a+ib)(c+id)=ac+iad+ibc+i^2bd$$

Since $i^2=-1$, this simplifies to:

$$z_1z_2=(ac-bd)+i(ad+bc)$$

Notice that the real part of z_1z_2 is $ac-bd$, and the imaginary part is $ad+bc$.

Example:

For $(2+i)(-5-4i)$:

$$(2+i)(-5-4i)$$
$$= 2(-5) + 2(-4i) + i(-5) + i(-4i)$$
$$= -10 - 8i - 5i - 4i^2$$

Since $i^2 = -1$

$$-10 - 8i - 5i + 4 = -10 - 13i + 4 = -6 - 13i$$

Commutativity:

Multiplication of complex numbers is commutative:

$$z_1z_2 = z_2z_1$$

For example, $(-5-4i)(2+i) = -6-13i$, just as shown previously.

Example 1.6: Multiply $(2+3i)(4+7i)$.

Solution:

$(2+3i)(4+7i) = (2)(4) + (2)(7i) + (4)(3i) + (3i)(7i)$

$= 8 + 14i + 12i + 21(-1)$ $\quad (\because i^2 = -1)$

$= (8-21) + (14+12)i$

$= -13 + 26i$.

Note: When performing operations with square roots of negative numbers, begin by expressing all square roots in terms of i. Then perform the indicated operation.

Incorrect:

$$\sqrt{-25}\cdot\sqrt{-4} = \sqrt{(-25)(-4)}$$
$$= \sqrt{100} = 10$$

Correct:

$$\sqrt{-25}\cdot\sqrt{-4} = i\sqrt{25}\cdot i\sqrt{4}$$
$$= 5i\cdot 2i$$
$$= 10i^2 = 10(-1) = -10$$

(iv) Division of Complex Numbers:

Dividing one complex number by another directly is challenging because the denominator contains two distinct terms. To overcome this, we multiply both the numerator and the denominator by the conjugate of the denominator. This process, known as rationalization , simplifies the expression by eliminating the imaginary component in the denominator, making the division possible.

We have $\dfrac{z_1}{z_2} = \dfrac{a_1 + b_1 i}{a_2 + b_2 i}$

$$= \frac{a_1 + b_1 i}{a_2 + b_2 i} \times \frac{a_2 - b_2 i}{a_2 - b_2 i} \quad \text{(by rationalization)}$$

$$= \frac{(a_1 + b_1 i)}{(a_2 + b_2 i)} \times \frac{(a_2 - b_2 i)}{(a_2 - b_2 i)}$$

$$= \frac{(a_1a_2 + b_1b_2) - (a_1b_2 - b_1a_2)i^2}{a_2^2 + b_2^2}$$

$$= \frac{(a_1a_1 + b_1b_2) - (a_1b_2 - b_1a_2)i}{a_2^2 + b_2^2} \quad (\because i^2 = -1)$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + \frac{b_1a_2 - a_1b_2}{a_2^2 - b_2^2} i$$

Thus $\boxed{\dfrac{z_1}{z_2} = \dfrac{a_1 + b_1 i}{a_2 + b_2 i} = \dfrac{a_1a_2 + bd}{a_2^2 + b_2^2} + \dfrac{bc - ad}{a_2^2 + b_2^2} i}$

Example 1.7: Write $\dfrac{2+3i}{3-5i}$ in the form $a + bi$.

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Solution: $\dfrac{2+3i}{3-5i}\times\dfrac{3+5i}{3+5i}$

$$=\frac{(2+3i)(3+5i)}{(3-5i)(3+5i)}$$

$$=\frac{6+10i+9i+15(i)^2}{9+15i-15i-25(i)^2}$$

$$=\frac{6+19i+15(-1)}{9-25(-1)}$$

$$=\frac{-9+19i}{34}=-\frac{9}{34}+\frac{19}{34}i$$

Skill 1.1

- **Understanding Complex Numbers:** Proficiency in representing complex numbers as $z=a+bi$, where a and b are real, and i is the imaginary unit.
- **Finding Complex Conjugates:** Ability to find the conjugate of $z=a+bi$ as $\bar{z}=a-bi$.
- **Calculating Modulus:** Skill in calculating the modulus $|z| = \sqrt{a^2+b^2}$, indicating its distance from the origin on the complex plane.

Exercise 1.1

1. Simplify and write the complex number as i, $-i$, -1 and 1.

(i) $-i^{40}$ **(ii)** i^{223} **(iii)** i^{2001}
(iv) i^0 **(v)** i^{-1}

2. Add the following complex numbers.

(i) $4(2+3i), -3(1-2i)$

(ii) $\frac{1}{3}-\frac{2}{3}i, \frac{1}{2}-\frac{1}{4}i$

(iii) $(\sqrt{3},1),(1,\sqrt{3})$

(iv) $\left(\frac{4}{5},\frac{\sqrt{3}}{4}\right),\left(\frac{\sqrt{3}}{4},\frac{4}{5}\right)$

3. Subtract the following complex numbers.

(i) $2\sqrt{2}-5\sqrt{7}i$ from $5\sqrt{2}-9\sqrt{7}i$

(ii) $\left(-7,\frac{1}{3}\right)$ from $\left(7,\frac{1}{3}\right)$

(iii) $(x, 0)$ from $(3, -y)$

(iv) $2x-3yi$ from $4x-7yi$

4. Multiply the following complex numbers:

(i) $(8i+11)(-7+5i)$

(ii) $(5i)(1-2i)$

(iii) $(9-12i)(15i+7)$

5. Perform the indicated division and write the answer in the form $a+bi$.

(i) $\frac{4+i}{3+5i}$ **(ii)** $\frac{1}{-8+i}$

(iii) $\frac{1}{7-3i}$ **(iv)** $\frac{6+i}{i}$

6. Prove that the sum as well as product of Complex numbers and its conjugate is a real number.

7. Write each expression as a complex number in the form $z=a+bi$.

(i) $(1-i)-2(4+i)^2$ **(ii)** $(1-i)^3$

(iii) $(2i)(8i)$ **(iv)** $(-6i)(-5i)^2$

8. Find the indicated absolute value of each complex number.

(i) $|3+4i|$ **(ii)** $|8-5i|$

9. If $z_1 = 3+2i$ and $z_2 = 4+5i$, then evaluate.

(i) $|z_1+z_2|$ **(ii)** $|z_1-z_2|$

(iii) $|z_1|$ **(iv)** $\left|\frac{z_1}{z_2}\right|$

10. Simplify and write your answer separately into real and imaginary parts.

(i) $\frac{2+3i}{5-2i}$ **(ii)** $\frac{(1+2i)^2}{1-3i}$ **(iii)** $\frac{1-i}{(1+i)^2}$

11. Show that $z.\bar{z}$ is a real number.

12. Show that $z=\bar{z}$ if z is real.

Student Learning Outcomes

- Demonstrate additive identity and multiplicative identity for the set of complex numbers
- Find additive inverse and multiplicative inverse of a complex number z

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1.5 Properties of Complex Numbers

This section focuses on the distinctive properties of complex numbers, examining their algebraic rules and behaviors. We will explore how these numbers, comprising real and imaginary components, operate within the framework of advanced mathematics

1.5.1 Properties of complex numbers with respect to addition and multiplication

Properties of addition and multiplication in complex numbers also holds. Like the addition and multiplication properties of real numbers. Now, look at the following properties.

1. Closure Property:

Addition: When you add two complex numbers together, you always get another complex number. For example, if you add $2 + 3i$ and $4 + 5i$, the result is $6 + 8i$, which is still a complex number.

Multiplication: When you multiply two complex numbers together, the result is always another complex number. For instance, multiplying $1 + 2i$ by $3 + 4i$ gives $-5 + 10i$, which is also a complex number.

2. Commutative Property:

Addition: The order in which you add two complex numbers doesn't matter. For example, $2 + 3i + 4 + 5i$ is the same as $4 + 5i + 2 + 3i$. You get the same result either way.

Multiplication: The order in which you multiply two complex numbers doesn't matter either. For example, multiplying $1 + 2i$ by $3 + 4i$ gives the same result as multiplying $3 + 4i$ by $1 + 2i$.

3. Associative Property:

Addition: The way you group complex numbers when adding them doesn't change the result. For example, if you add $(1 + 2i) + (3 + 4i)$ first, then add $5 + 6i$, it's the same as adding $1 + 2i$ to $(3 + 4i + 5 + 6i)$.

Multiplication: Similarly, the way you group complex numbers when multiplying them doesn't change the result. For example, $(1 + 2i) \times (3 + 4i) \times (5 + 6i)$ gives the same answer no matter which two you multiply first.

4. Distributive Property:

Multiplication over Addition: If you multiply a complex number by a sum of two others, it's the same as multiplying the first complex number by each of the others separately, and then adding the results. For example, $2 \times (3 + 4i + 5 + 6i)$ is the same as $2 \times (3 + 4i) + 2 \times (5 + 6i)$.

5. Additive identity and multiplicative identity of Complex Numbers

This topic introduces the foundational concepts of additive and multiplicative identities in the context of complex numbers. We will define the additive identity, which leaves a number unchanged in addition, and the multiplicative identity, which preserves a number during multiplication, and explore their applications in complex number arithmetic

Additive Identity

A complex number $a_2 + b_2i$ is called the **additive identity** of the complex number $a_2 + b_2i$ if

$(a_1 + b_1i) + (a_2 + b_2i) = a_1 + b_1i$

Let $a_1 + b_1i$ be any complex number and $a_2 + b_2i = 0 + 0i$ be the zero complex number. Then

$(a_1 + b_1i) + (0 + 0i) = (a_1 + 0) + (b_1 + 0)i$ (by definition of addition)

$$= a_1 + b_1i$$

Similarly, $(0 + 0i) + (a_1 + b_1i) = a_1 + b_1i$

Therefore, the additive identity in C is the zero complex number i.e.

$$\boxed{0 + 0i}$$

Multiplicative Identity

A complex number $1 + 0i$ is called the **multiplicative identity** of the complex number $a_1 + b_1i$ if

$(a_1 + b_1i)(a_2 + b_2i) = (a_2 + b_2i)(a_1 + b_1i) = a_1 + b_1i$

Let $a_1 + b_1i$ be any complex number and $a_2 + b_2i = 1 + 0i$ be the unit complex number. Then

$(a_1 + b_1i)(1 + 0i) = (a_1 . 1 - b_1.0) + (a_1.0 + b_1.1)i$ (by definition of multiplication)

$$(a_1 + b_1i)(1 + 0i) = (a_1, b_1)(1, 0)$$

$$= (a_1 - 0, 0 + b_1) \quad \text{(by × property)}$$

$$= (a_1, b_1) = a_1 + b_1i$$

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Similarly, $(1 + 0i)(a_1 + b_1i) = a_1 + b_1i$

Thus the multiplicative identity in C is the unit complex number i.e. $\boxed{1+0i}$

6. Additive inverse and multiplicative inverse of complex numbers.

Explore the concepts of additive and multiplicative inverses in complex numbers, focusing on how the additive inverse neutralizes a number through addition, and the multiplicative inverse produces unity through multiplication. This understanding is key for effective manipulation of complex numbers

Additive Inverse

A complex number $-a_1 - b_1i$ is called the **additive inverse** of the complex number $a_1 + b_1i$ if

$$(a_1 + b_1i) + (-a_1 - b_1i) = (a_1, b_1) + (-a_1, -b_1)$$

$$= (a_1 - a_1, b_1i - b_1)$$

$$= (0, 0) \text{ additive property}$$

Therefore,

$\boxed{\text{the additive inverse of } a_1 + b_1i \text{ is } -a_1 - b_1i}$

Example 1.8: Find additive inverse of $8 - 5i$

Solution:

Let $x + yi$ be the inverse of $8 - 5i$ then by definition

$(8 - 5i) + (x + yi) = 0 + 0i$

$(8 + x) + (-5 + y)i = 0 + 0i$

$\Rightarrow \quad 8 + x = 0$ and $-5 + y = 0$

$\Rightarrow \quad x = -8$ and $y = 5$

$\because \quad x + yi = -8 + 5i$

Hence, the additive inverse of $8 - 5i$ is $-8 + 5i$

Multiplicative Inverse

A complex number $a_2 + b_2i$ is called the **multiplicative inverse** of the complex number $a_1 + b_1i$ if

$(a_1 + b_1i)(a_2 + b_2i) = 1 + 0i$ i.e. the multiplicative identity.

We have $(a_1 + b_1i)(a_2 + b_2i) = 1 + 0i$

$\Rightarrow \quad (a_1a_2 - b_1b_2) + (a_2b_2 + b_1b_2)i = 1 + 0i$

$\Rightarrow \quad a_1a_2 - b_1b_2 = 1 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$

and $\quad a_1b_2 + b_1a_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$

From (ii), we have

$a_1b_2 = -b_1a_2$ or $b_2 = \dfrac{-b_1a_2}{a_1} \quad \ldots\ldots\ldots\ldots\ldots$ (iii)

Putting the value of b_2 in (i), we get

$a_1a_2 + b_1\dfrac{b_1a_2}{a_1} = 1 \quad \Rightarrow \quad \dfrac{a_1^2a_2 + b_1^2a_2}{a_1} = 1$

$\Rightarrow \quad (a_1^2 + b_1^2)a_2 = a_1$

$\Rightarrow \quad a_2 \dfrac{a_1}{a_1^2 + b_1^2} \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$

Putting the value of a_2 in (iii), we get

$$b_2 = \frac{-b_1.a_1}{a_1(a_1^2 + b_1^2)}$$

$\Rightarrow \quad b_2 = \dfrac{-a_1}{a_1^2 + b_1^2} \quad \ldots\ldots\ldots\ldots\ldots \text{(v)}$

From (iv) and (v), we have

$$\boxed{a_2 + b_2i = \frac{a_1}{a_1^2 + b_1^2} - \frac{b_1}{a_1^2 + b_1^2}i}$$

Thus the multiplicative inverse of

$a_1 + b_1i$ is $\dfrac{a_1}{a_1^2 + b_1^2} - \dfrac{b_1}{a_1^2 + b_1^2}i$

Example 1.9: Find multiplicative inverse of $4 + 7i$

Solution: Let $x + yi$ be the multiplicative inverse of $4 + 7i$. Then by definition

$(4 + 7i)(x + yi) = 1 + 0i$

$4(x + yi) + 7i(x + yi) = 1 + 0i$

$4x + 4yi + 7xi + 7yi^2 = 1 + 0i$

$4x + (7x + 4y)i + 7y(-1) = 1 + 0i \quad (\because i^2 = -1)$

$(4x - 7y) + (7x + 4y)i = 1 + 0i$

$\Rightarrow \quad 4x + 7y = 1 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$

and $\quad 7x + 4y = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$

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Multiply equation (i) by 4 and equation (ii) by 7. Then add both equation.

$$16x - 28y = 4$$
$$49x + 28y = 0$$

$$65x = 4 \quad \text{or} \quad x = \frac{4}{65}$$

Now, putting the value of x in equation (i).

$$4\left(\frac{4}{65}\right) - 7y = 1 \qquad \frac{16}{65} - \frac{65}{65} = 7y$$

$$\frac{16}{65} - 7y = 1 \qquad \frac{49}{65} = 7y$$

$$\frac{16}{65} - 1 = 7y \qquad \Rightarrow \quad y = \frac{7}{65}$$

$$\because \quad x + yi = -\frac{4}{65} + \frac{7}{65}i$$

Multiplicative inverse of $4 + 7i$ is $\frac{4}{65} - \frac{7}{65}i.$

Thus $-\frac{2}{13} + \frac{3}{13}i$ is the multiplicative inverse of $-2 - 3i$

> The complex numbers possess all the properties that real numbers possess except for the order relation, that is, we cannot say that one complex number is greater than the other complex number.

Student Learning Outcomes

- Demonstrate the following properties of a complex number z

$$|z| = |-z| = |\bar{z}| = |-\bar{z}|,$$

$$\bar{\bar{z}} = z, \; z\bar{z} = |z|^2, \bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$$

$$\overline{z_1 z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{\left(\frac{z_1}{z}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$$

1.6 Some Properties of the Conjugate and Modulus of Complex Numbers

In the following theorem we prove some properties pertaining to conjugation and modulus of complex numbers.

Theorem 1: For all z_1, z_2, z_3 in C

(i) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$ (ii) $\bar{\bar{z}} = z$

(iii) $z\bar{z} = |z|^2$ (iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

Proof: **(i)** Let $z = a_1 + b_1 i$. Then

$-z = -a_1 - b_1 i$, $\bar{z} = a_1 - b_1 i$ and

$-\bar{z} = -a_1 + b_1 i$

Therefore by definition

$$|z| = \sqrt{a_1^2 + b_1^2} \quad \text{\_\_\_\_\_\_\_ (i)}$$

$$|-z| = \sqrt{(-a_1^2) + (-b)_1^2} = \sqrt{a_1^2 + b_1^2} \quad \text{\_\_\_\_\_\_\_ (ii)}$$

$$|\bar{z}| = \sqrt{a_1^2 + (-b_1^2)} = \sqrt{a_1^2 + b_1^2} \quad \text{\_\_\_\_\_\_\_ (iii)}$$

$$|-\bar{z}| = \sqrt{(-a_1)^2 + (b_1)^2} = \sqrt{a_1^2 + b_1^2} \quad \text{\_\_\_\_\_\_\_ (iv)}$$

Equation (i), (ii), (iii) and (iv) yield that

$$|z| = |-z| = |\bar{z}| = |-\bar{z}|$$

(ii) Let $z = a_1 + b_1 i$, then $\bar{z} = \overline{a_1 + b_1 i} = a_1 - b_1 i$ and

so $\bar{\bar{z}} = \overline{a_1 + b_1 i} = a_1 - b_1 i = z$

Thus $\bar{\bar{z}} = z$

(iii) Let $z = a_1 + b_1 i$. Then $\bar{z} = a_1 - b_1 i$

Therefore $z\bar{z} = (a_1 + b_1 i)(a_1 - b_1 i)$

$$= a_1^2 - a_1 b_1 i + b_1 a_1 i - b_1^2 i^2$$

$$= a_1^2 - (-1) b_1^2 \qquad (\because i^2 = -1)$$

$$= a_1^2 + b_1^2$$

$$= |z|^2 \qquad (\because |z| = \sqrt{a_1^2 + b_1^2})$$

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Thus $\quad \bar{z}z = |z|^2$

(iv) Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Then $\bar{z}_1 = a_1 - b_1 i$, $\bar{z}_2 = a_2 - b_2 i$ and

$$z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$

$$= (a_1 + a_2) + (b_1 + b_2)i$$

Therefore $\overline{z_1 - z_2} = (a_1 + a_2) - (b_1 + b_2)i$

$$= (a_1 - b_1 i) + (a_2 - b_2 i)$$

$$= \bar{z}_1 + \bar{z}_2$$

Thus $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Then $\overline{z_1 z_2} = \overline{(a_1 + b_1 i)(a_2 + b_2 i)}$

$$= \overline{(a_1 a_2 + b_1 b_2) + (a_1 b_2 + b_1 a_2)i}$$

$$= (a_1 a_2 - b_1 b_2) - (a_1 b_2 + b_1 a_2)i \quad \text{\_\_\_\_\_\_ (i)}$$

and $\bar{z}_1 \ \bar{z}_2 = \overline{(a_1 + b_1 i)} \ \overline{(a_2 + b_2 i)}$

$$= (a_1 - b_1 i)(a_2 - b_2 i)$$

$$= (a_1 a_2 - b_1 b_2) + (-ad - bc)i$$

$$= (a_1 a_2 - b_1 b_2) - (a_1 b_2 + b_1 a_2)i \quad \text{\_\_\_\_\_\_ (ii)}$$

Thus from equations (i) and (ii) we have

$$\overline{z_1 z_2} = \bar{z}_1 \,.\, \bar{z}_2$$

(vi) Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Then $\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$

$$= \frac{a_1 + b_1 i}{a_2 + b_2 i} \times \frac{a_2 - b_2 i}{a_2 - b_2 i} \quad \text{(by rationalization)}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + (b_1 a_2 - a_1 b_2)i}{a_2^2 + b_2^2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} - \frac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2} i$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2} i \quad \text{\_\_\_\_\_\_ (i)}$$

and $\frac{\bar{z}_1}{\bar{z}_2} = \frac{\overline{a_1 + b_1 i}}{\overline{a_2 + b_2 i}}$

$$= \frac{a_1 - b_1 i}{a_2 - b_2 i}$$

$$= \frac{a_1 - b_1 i}{a_2 - b_2 i} \times \frac{a_2 + b_2 i}{a_2 + b_2 i} \quad \text{(by rationalization)p}$$

$$= \frac{(a_1 a_2 + b_1 b_2) - (b_1 a_2 - a_1 b_2)i}{a_2^2 + b_2^2} \quad \text{\_\_\_\_\_\_ (ii)}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2} i \quad \text{\_\_\_\_\_\_ (ii)}$$

Thus from equations (i) and (ii), we have

$$\overline{\left(\frac{z_1}{z^2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Find real and imaginary parts of the following types of complex numbers.

Where n = ± 1 and ± 2

1.7 Real and Imaginary Parts of the Complex Numbers of the form

Complex numbers, expressed as $x+iy$, contain real and imaginary parts. This section focuses on extracting these parts from complex expressions like $(x+iy)^n$ and more intricate forms involving division and exponentiation.

(i) $(x+iy)^n$ (ii) $\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^n$ where $n = \pm 1$ and ± 2

Solution:

1. Real and imaginary parts of $(x + iy)^n$ where $n = \pm 1$ and ± 2

When $n = 1$, $(x + iy)^n$ reduces to $x + iy$

Therefore,

Real part = x and imaginary part = y

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when $n = -1$, $(x+iy)^n$ reduces to $(x+iy)^1$

We have $(x + iy)^{-1} = \dfrac{1}{(x+iy)}$

$$= \frac{1}{(x+iy)} \times \frac{x-iy}{x-iy} \quad \text{(by rationalization)}$$

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{(x+iy)} - i\frac{y}{x^2-y^2}$$

Therefore,

$$\text{Real part} = \frac{x}{x^2+y^2}$$

$$\text{Imaginary part} = \frac{-y}{x^2+y^2}$$

When $n = 2$, $(x+iy)^n$ reduces to $(x+iy)^2$,

we have $(x+iy)^2 = x^2 + 2ixy + i^2y^2$

$$= x^2 + 2ixy - y^2$$

$(\because i^2 = -1)$

$$= (x^2 - y^2) + 2ixy$$

Therefore,

$$\text{Real part} = x^2 - y^2$$

$$\text{Imaginary part} = 2xy$$

When $n = -2$, $(x+iy)^n$ reduces to $(x+iy)^2$

We have $(x+iy)^2 = \dfrac{1}{(x+iy)^2}$

$$= \frac{1}{(x+iy)^2} \times \frac{(x-iy)^2}{(x-iy)^2}$$

$$= \frac{x^2-y^2-2ixy}{[(x+iy)(x-iy)]^2}$$

$$= \frac{x^2-y^2-2ixy}{(x^2+y^2)^2} \quad = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2}$$

Therefore,

$$\text{Real part} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\text{Imaginary part} = \frac{-2xy}{(x^2+y^2)^2}$$

Example 1.10: Find the real and imaginary parts of the following complex numbers.

(i) $1 + 8i$ **(ii)** $(3 - 5i)^{-1}$

(iii) $(7 + i)^2$ **(iv)** $(5 - 2i)^{-2}$

Solution:

(i) Let $z = 1 + 8i$ where $x = 1, y = 8$

Therefore real part of $z = 1$

Imaginary part of $z = 8$

(ii) Let $z = (3 - 5i)^{-1}$,

Here $x = 3$ and $y = -5$

Therefore,

Real part of $z = \dfrac{x}{x^2+y^2} = \dfrac{3}{(3)^2+(-5)^2} = \dfrac{3}{9+25} = \dfrac{3}{34}$

Imaginary part of $z = \dfrac{-y}{x^2+y^2} = \dfrac{-(-5)}{(3)^2+(-5)^2} = \dfrac{5}{9+25}$

$$= \frac{5}{34}$$

(iii) Let $z = (7 - i)^2$ Here, $x = 7$, $y = -1$. Therefore,

Real part of $z = x^2 - y^2 = (7)^2 - (1-1)^2 = 49 - 1 = 48$

Imaginary part of $z = 2xy = 2(7)(-1) = -14$

(iv) Let $z = 5 - 2i$. Where $x = 5$, $y = -2$

Therefore,

Real part of $z = \dfrac{x^2-y^2}{(x^2+y^2)^2} = \dfrac{(5)^2-(-2)^2}{[(5)^2+(-2)^2]^2}$

$$= \frac{25-4}{(25+4)^2} = \frac{21}{841}$$

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Imaginary part of $z =$

$$\frac{-2xy}{(x^2+y^2)^2} = \frac{-2(5)(-2)}{[(5)^2+(-2)^2]^2} = \frac{20}{(25+4)^2} = \frac{20}{841}$$

2. Real and imaginary parts of $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$ where $n = \pm 1$ and ± 2

When $n = 1$, $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\frac{x_1+iy_1}{x_2+iy_2}$.

We have $\frac{x_1+iy_1}{x_2+iy_2}$

$$= \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2+iy_2} \text{ (by rationalization)}$$

$$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 - i^2y_1y_2}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2 - i(x_1y_2 + y_1x_2) + y_1y_2}{x_2^2+y_2^2} \qquad (\because i^2 = -1)$$

$$= \frac{(x_1x_2+y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\frac{y_1y_2 - x_1x_2}{x_2^2+y_2^2}$$

Therefore,

Real part $= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}$

Imaginary part $= \frac{y_1x_2 - x_1y_2}{x_2^2+y_2^2}$

Therefore,

Real part $= \frac{(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1x_2y_1y_2}{(x_2^2+y_2^2)^2}$

Imaginary part $= 2\frac{x_1y_1(x_2^2-y_2^2) - x_2y_2(x_1^2-y_1^2)}{(x_2^2+y_2^2)^2}$

When $n = -1$, $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-1}$

We have $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-1} = \frac{x_2+iy_2}{x_1+iy_1}$

$$= \frac{x_2+iy_2}{x_1+iy_1} \times \frac{x_1+iy_1}{x_1+iy_1} \text{ (by rationalization)}$$

$$= \frac{x_2x_1+y_2y_1}{x_1^2+y_1^2} + i\frac{y_2x_1 - x_2y_1}{x_1^2+y_1^2}$$

Therefore,

Real part $= \frac{x_2x_1+y_2y_1}{x_1^2+y_1^2}$

Imaginary part $= \frac{y_2x_1 - x_2y_1}{x_1^2+y_1^2}$

When $n = 2$, $\left(\frac{x_1-iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\frac{x_1-iy_1}{x_2+iy_2}\right)^2$

We have $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^2 = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2}$

$$= \frac{(x_1-iy_1)^2}{(x_2+iy_2)^2} \times \frac{(x_2-iy_2)^2}{(x_2+iy_2)^2} \text{ (by rationalization)}$$

$$= \frac{\left[(x_1^2-y_1^2)+2ix_1y_1\right]\left[(x_2^2-y_2^2)-2ix_2y_2\right]}{\left[(x_2+iy_2)(x_2-iy_2)\right]^2}$$

$$= \frac{(x_1^2-y_1^2)(x_2^2-y_2^2)+4x_1x_2y_1y_2}{(x_2^2+y_2^2)^2} + \frac{2i\left[x_1y_1(x_2^2-y_2^2) - x_2y_2(x_1^2-y_1^2)\right]}{(x_2^2+y_2^2)^2}$$

When $n = -2$, $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)$ reduces to $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-2}$

We have $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^{-2} = \frac{(x_2+iy_2)^2}{(x_1+iy_1)^2}$

$$= \frac{(x_2+iy_2)^2}{(x_1+iy_1)^2} \times \frac{(x_1-iy_1)^2}{(x_1-iy_1)^2}$$

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$$=\frac{\left[\left(x_2^2-y_2^2\right)\left(x_1^2-y_1^2\right)+4x_1x_2y_1y_2\right]+2i\left[x_2y_2\left(x_1^2-y_1^2\right)-x_1y_1\left(x_2^2-y_2^2\right)\right]}{\left(x_1^2+y_1^2\right)^2}$$

Therefore,

$$\text{Real part}=\frac{\left(x_2^2-y_2^2\right)\left(x_1^2-y_1^2\right)+4x_1x_2y_1y_2}{\left(x_1^2+y_1^2\right)^2}$$

$$\text{Imaginary part}=2\frac{x_2y_2\left(x_1^2-y_1^2\right)-x_1y_1(x_2^2-y_2^2)}{\left(x_1^2+y_1^2\right)^2}$$

Example 1.11: Find the real and imaginary parts of $\left(\frac{2+3i}{4+5i}\right)^{-2}$

Solution:

Let $z_1 = 2 + 3i$ where $x_1 = 2$, $y_1 = 3$

And $z_2 = 4 + 5i$ where $x_2 = 4$, $y_2 = 5$

$$\text{Real part of } \frac{z_1}{z_2}=\frac{(x_1{}^2-y_1{}^2)(x_2{}^2-y_2{}^2)+4x_1x_2.y_1y_2}{(x_1^2+y_1^2)^2}$$

$$=\frac{[(2)^2-(3)^2][(4)^2-(5)^2]+4(2)(4)(3)(5)}{[(2)^2+(3)^2]^2}$$

$$=[4-9][16-25]+480=\frac{525}{169}$$

Imaginary part of =

$$\frac{z_1}{z_2}=2\left[\frac{x_2y_2(x_1^2-y_1^2)-x_1y_1(x_2^2-y_2^2)}{(x_1^2+y_1^2)^2}\right]$$

$$=2\left[\frac{(4)(5)[(2)^2-(3)^2]-(2)(3)[(4)^2-(5)^2]}{[(2)^2-(3)^2]}\right]$$

$$=2\left[\frac{20(-5)-6(-9)}{169}\right]$$

$$=2\left[\frac{-100+54}{169}\right]$$

$$=2\left[\frac{-46}{169}\right]=\frac{-92}{169}$$

Skill 1.2

- **Complex Number Properties:** Applying distributive, associative, identity, inverse, and commutative properties to complex numbers.
- **Conjugate Characteristics:** Understanding and using the conjugate of a complex number **z.**
- **Complex Powers:** Analyzing $(x+iy)^n$ to extract real and imaginary parts for various Powers of n.

Exercise 1.2

1. If $z_1 = 1 - 5i$ and $z_2 = 4 - 5i$ name and verify the following properties

(i) $z_1 + z_2 = z_2 + z_1$ **(ii)** $z_1 z_2 = z_2 z_1$

2. $z_1 = -6 + i$, $z_2 = 3 - 2i$ and $z_3 = 2 + 3i$ name and verify the following properties

(i) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

(ii) $z_1 (z_2 z_3) = (z_1 z_2) z_3$

What properties are these?

3. If $z_1 = 5+3i, z_2 = 7-2i$ and $z_3 = 3-2i$ name and verify

the following properties $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

4. Find the additive inverse of the following complex numbers.

(i) $(\sqrt{2}, -\sqrt{5})$ **(ii)** $(4, -3)$ **(iii)** $\frac{2}{11}+\frac{3}{11}i$

5. Find the multiplicative inverse of the following complex numbers.

(i) $2 + i$ **(ii)** $(-1, 3)$ **(iii)** $7 + 9i$

6. Let $z_1 = 2 + 4i$ and $z_2 = 1 - 3i$. Verify for this z_1 and z_2 that $\overline{z_2+z_2}=\bar{z}_1+\bar{z}_2$.

7. Let $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$. Verify for this z_1 and z_2 that $\overline{z_2z_2}=\bar{z}_1\bar{z}_2$ and $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$.

8. Show that for all complex number z_1 and z_2.

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(i) $|z_1 z_2| = |z_1||z_2|$ **(ii)** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

where $z_2 \neq 0$.

10. Separate the following into real and imaginary parts the following complex numbers

(i) $4+i$ **(ii)** $(6-5i)^2$

(iii) $(4-7i)^{-1}$ **(iv)** $(2p-qi)^{-2}$

(v) $\frac{4-3i}{1-i}$ **(vi)** $\left(\frac{2-3i}{4+5i}\right)^{-1}$

(vii) $\left(\frac{\sqrt{2}-i}{\sqrt{2}+i}\right)^2$ **(viii)** $\left(\frac{1+\sqrt{2}i}{1-\sqrt{2}i}\right)^{-2}$

Student Learning Outcomes

✧ Solve the simultaneous linear equations with complex coefficients

1.8 Solution of equations

To find the solution of different equations in complex variables either with real or complex co- efficient, we use some techniques which we used to find the solution of simultaneous linear equations.

1.9.1 Solution of Simultaneous Linear Equations with Complex Co-efficients

This topic explores the resolution of simultaneous linear equations with complex coefficients. Interestingly, solving a complex linear system closely parallels the process of solving a system of two linear equations. We will examine the methods and principles that underlie this approach

Consider the following equation

$$\ell z + mw = n \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where ℓ, m and n are complex numbers. The equation (i) is called a **linear equation** in two complex variables (or unknown) z and w.

$$\ell_1 z + m_1 w = n_1$$
$$\ell_2 z + m_2 w = n_2 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

These two equations together form a system of linear equations in two variables z and w.

The linear equations in two variables are also known as **simultaneous linear equations**.

For example

$$\begin{Bmatrix} 5z-(3+i)w = 7-i \\ (2-i)z+2iw = -1+i \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

is a system of linear equations with complex co–efficients.

A **solution** of a system in two variables z and w is an ordered pair (z,w) such that both the equations in the system are satisfied. For example consider system (iii). The ordered pair (z,w) where $z = 1+i$ and $w = 2i$ is a solution of (iii) because if we replace z by $1+i$ and w by $2i$, then both the equations are satisfied. The process of finding all solutions of the system of equations is known as **solving the system**.

Here we shall find solution of a system of two equations with complex co-efficients in two variables z and w. The simple rule for solving such system of equations is the "Method of **Elimination and Substitution**".

(i) If necessary, multiply each equation by a constant so that the co-efficient of one variable in equation is the same.

(ii) Add or subtract the resulting equations to eliminate one variable, thus getting an equation in one variable.

(iii) Solve the equation in one variable obtained in step-2.

(iv) Put the known value of one variable in either of the original equation in step-1 and solve for the other variable.

(v) Writing together the corresponding values of the variables in the form of ordered pairs gives solution of the system.

Example 1.12: Solve the simultaneous linear equations with complex co-efficients.

$$5z-(3+i)w = 7-i$$
$$(2-i)z+2iw = -1+i$$

Solution: Since,

$$5z-(3+i)w = 7-i \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

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$(2-i)z + 2iw = -1 + i$ …………… (ii)

Multiplying equation (i) by $(2-i)$ we have

$5(2-i)z - (3+i)(2-i)w = (7-i)(2-i)$

$\Rightarrow 5(2-i)z - (6 - 3i + 2i - i^2)w = 14 - 7i - 2i + i^2$

$\Rightarrow 5(2-i)z - (6 - i + 1)w = 14 - 9i - 1 \quad (\because i^2 = -1$

$\Rightarrow 5(2-i)z - (7-i)w = 13 - 9i$ …… (iii)

Multiplying equation (ii) by 5, we have

$5(2-i)z + 10iw = -5 + 5i$ …… (iv)

Subtracting equation (iii) from equation (iv), we have

$$\begin{array}{rl} 5(2-i)z + 10iw & = -5 + 5i \\ -5(2-i)z \mp (7-i)w & = -13 \mp 9i \\ \hline 10iw + (7-i)w & = -18 + 14i \end{array}$$

$\Rightarrow (7+9i)w = -18 + 14i$

$$\Rightarrow w = \frac{-18+14i}{7+9i}$$

$$\Rightarrow w = \frac{-18+14i}{7+9i} \times \frac{7-9i}{7-9i} \quad \text{(by rationalization)}$$

$$\Rightarrow w = \frac{260i}{130} = 2i$$

By putting the value of w in (1), we have

$5z - (3+i)(2i) = 7 - i$

$\Rightarrow 5z - (6i + 2i^2) = 7 - i$

$\Rightarrow 5z - (6i - 2) = 7 - i$

$\Rightarrow 5z = 7 - i + 6i - 2$

$\Rightarrow 5z = 5 + 5i$

$$\Rightarrow z = \frac{5+5i}{5} = 1 + i$$

Thus (z, w) where $z = 1 + i$ and $w = 2i$ is the solution of the simultaneous linear equations.

Skill 1.3

Complex Linear Equation Solution:
Mastering the solution of linear equations with complex coefficients

Exercise 1.3

1. Solve the simultaneous linear equations with complex co-efficient.

(i) $z + 3w = 5i$, $3z + 4w = 2$

(ii) $4z - 2w = 2i + 7$, $z + 2w = 10 - 3i$

(iii) $z + (2+i)w = 15i$, $(4-i)z - 3w = 3i$

(iv) $2z + w = 4 + i$, $6z - w = 8 + 2i$

(v) $z - iw = 3 + 2i$, $2z + (1-3i)w = 5i$

(vi) $(3+2i)z + w = 7$, $z - 4w = 1 - i$

1.9 Mathematical Modelling

Mathematical modelling is a technique using which we can represent a physical system by an equation or a set of equations. Once an equation or a set of equations has been developed, they must be solved to obtain results. These results can provide us with useful information regarding the system. This technique is widely used in all science fields such as physics, chemistry, and biology to predict the behaviors of physical systems. Mathematical models are mostly built around theoretical principles or hypotheses. A mathematical model built upon a theory or hypothesis is often used to obtain numerical data which can be compared with experimental results to validate the hypothesis.

Using mathematical models in physics, we can model how planets move in space. In chemistry, we can predict what happens when different chemicals mix. In biology, we can understand how a population of an animal species changes over time. So, these models help us make sense of the world. They give us clues and help us solve the mysteries of how things work in the world around us.

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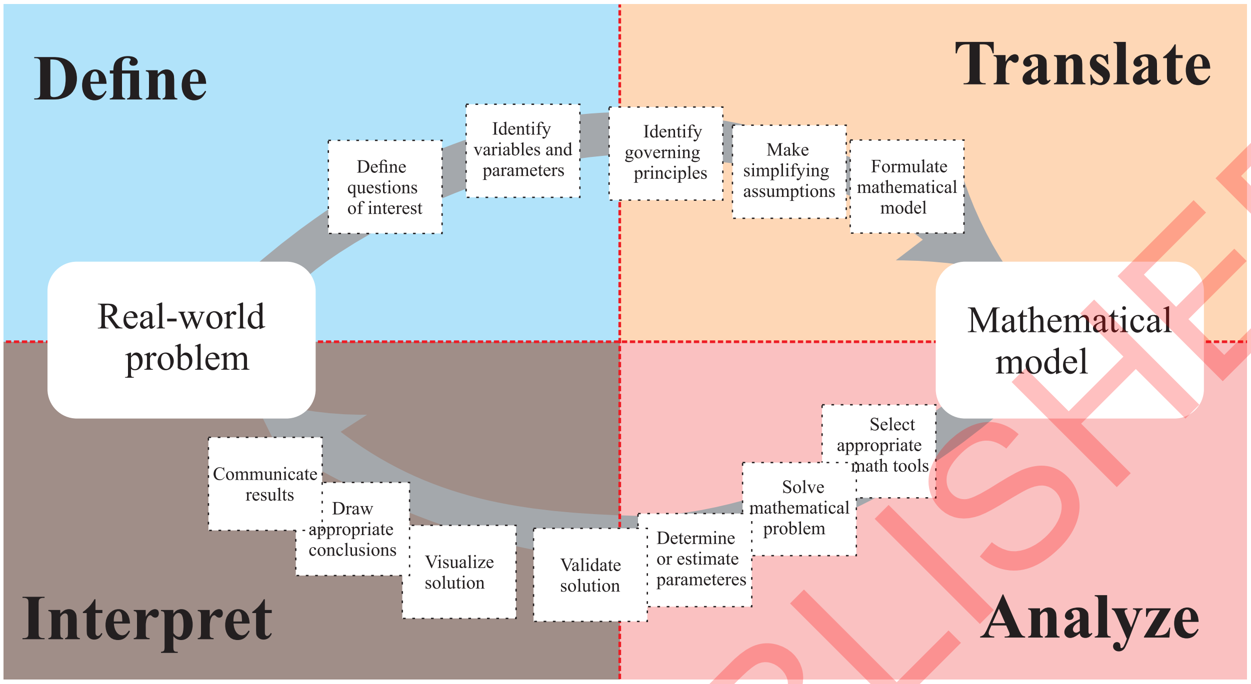

Transformation of a Daily life real world problem to a mathematical model.

Example 1.13:

Assume a ball of mass 200g is thrown up in the air with an initial velocity of 10 m/s. Calculate the maximum height it reaches 2s after leaving the ground.

Solution

This case deals with projectile motion. The height at any given time can be obtained by using the second equation motion: $h = v_i t + \frac{1}{2}gt^2$

We can factorize it to obtain:

Inserting the values of parameters provided

$$h = 10\left(10 + \frac{1}{2}(-9.8)(2)\right)$$

$$h = 20\,m$$

An interesting thing to note here is that the height attained by the ball is independent of its mass as we do not see mass in the mathematical model. This simple problem also illustrates that we can calculate the speed a particular object must be thrown to attain a particular height. This simple model can be used to understand the principle upon which projectiles work.

Example 1.14:

The total energy possessed by a system can be expressed as a sum of both kinetic and potential energies. Let an object of mass 0.2 kg be placed at a height of 100m with no initial velocity. Now the ball starts falling and about 5s after falling, the velocity of the object is given by 50 m/s. Calculate its total energy.

Hint: Follow Example 1 to calculate height.

Solution:

Firstly, we develop the equation for the height of the object: $h = v_i t + \frac{1}{2}gt^2$

$$h = (0)t + \frac{1}{2}gt^2$$

$$h = \frac{1}{2}gt^2$$

Now, we need to develop the equation for total energy.

$$K.E = \frac{1}{2}mv^2 \text{ and } P.E = mgh$$

$$Total\ Energy = Kinetic\ Energy + Potential\ Energy$$

$$Total\ Energy = \frac{1}{2}mv^2 + mgh$$

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Now, inserting the value of h that we have calculated earlier:

$$Total\ Energy = \frac{1}{2}mv^2 + mg\left(\frac{1}{2}gt^2\right)$$

$$Total\ Energy = \frac{1}{2}m\left(v^2 + g^2t^2\right)$$

Now, inserting the values provided:

$$Total\ Energy = \frac{1}{2}(0.2)\left(50^2 + (-9.8)^2(5)^2\right)$$

$$Total\ Energy = 490.1\ J$$

In this problem, it was observed that sometimes to create a mathematical model another one might be incorporated into it to attain a final model. In complex engineering problems, there are many parameters that come into play. Each of these parameters might be dependent on different simpler parameters. Therefore, it is common practice to combine different models in order to simplify a complex problem.

Example 1.15:

A population of rabbits starts with 2 individuals and doubles every month. What will be the population after an year? Use the exponential growth formula $P = P_o \times 2^t$ where P is the population after time t, P_o is the initial population, and t is the time in months.

Solution

The data provided in the problem statement is as follows:

$$P_o = 2$$

$$t = 1\ year = 12\ months$$

Now putting these values into the mathematical model provided, we get:

$$P = 2 \times 2^{12}$$

$$P = 8192$$

This problem shows how a single pair of animals if bred correctly can create a massive population of a single species. Although in real scenarios a lot of other factors come into play and a much complex mathematical model might be used, this simple problem still provides critical information for biologists which are working with endangered animals.

Example 1.16:

Suppose you have a balloon with a volume of 1.5 litres at a temperature of 25 degrees Celsius. If the balloon is heated to a new temperature of 75 degrees Celsius while the pressure remains constant, what will be the new volume of the balloon? Use the Charles law: $\frac{V_1}{T_1} = \frac{V_2}{T_2}$ where T_1 is the initial temperature, V_1 is the initial volume, T_2 is the final temperature and V_2 is the final volume.

Solution

The data provided in the problem statement is as follows:

$$V_1 = 1.5\ litres \quad T_1 = 25°C = 298\ K \quad T_2 = 75°C = 348\ K$$

Now putting these values into the mathematical model provided, we get:

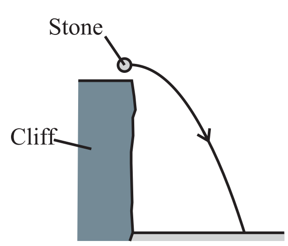
$$\frac{1.5}{298} = \frac{V_2}{348}$$

$$V_2 = 1.75\ litres$$

This problem demonstrates the behavior of gases. The balloon is merely a container for air which is the gas under observation in this problem. This exact principle is used in hot air balloons where the volume of the balloon is increased by increasing the temperature of the air trapped inside it.

Exercise 1.4

1. A stone is thrown horizontally off a cliff with an initial speed of 10 m/s. Calculate the time it takes for the stone to reach the ground. Use the equation $h = \frac{1}{2}gt^2$, where h is the distance fallen (height of the cliff), g is the acceleration due to gravity, and t is the time of fall.



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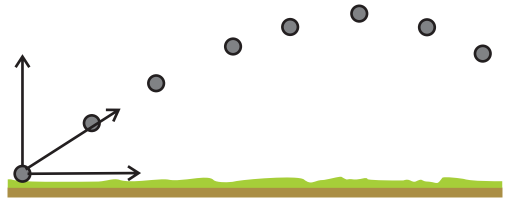
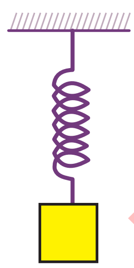
2. A projectile is launched with an initial velocity of 20 m/s at an angle of 30 degrees above the horizontal. Determine the maximum height reached by the projectile. Use the kinematic equation $h = \frac{v_y^2}{2g}$ where v_y is the vertical component of the initial velocity and g is the acceleration due to gravity.



3. A mass-spring system undergoes simple harmonic motion with a period of 2 seconds. Find the mass of the object attached to the spring if the spring constant is 100 N/m. The period of a mass-spring system is given by $T = 2\pi\sqrt{\frac{m}{k}}$ where T is the period, m is the mass, and k is the spring constant.



4. A student is conducting an experiment with a concave mirror. The mirror has a focal length of 15 cm. If an object is placed 30 cm in front of the mirror, determine the image distance and state whether the image is real or virtual. Hint: Use the mirror formula for concave mirrors, which is given by $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ where f is the focal length of the mirror, p is the object distance, and q is the image distance.

5. A population of rabbits starts with 50 individuals and doubles every year. How many years will it take for the population to reach 500. Use the exponential growth formula $P = P_o \times 2^t$ where P is the population after time t, P_o is the initial population, and t is the time in years.

6. A bacterial culture grows at a rate of 0.03 per minute. If the initial population is 100 bacteria, find the population after 20 minutes. Apply the exponential growth formula $P = P_o \times e^{rt}$ where P is the population after time t, P_o is the initial population, r is the growth rate and t is the time in minutes.

7. A population of rabbits in a forest grows at a rate of 10% per year. If the initial population is 200 rabbits, determine the population after 5 years. Use the formula for compound interest to model the exponential growth of the rabbit population: $P = P_o \times (1 + r)^t$ where P is the population after time, P_o is the initial population, r is the growth rate and t is the time in years.

8. A chemical reaction has a rate constant of 0.02 s^{-1}. Calculate the half-life of the reaction. The half-life ($t_{1/2}$) is related to the rate constant (k) by the equation: $t_{1/2} = \frac{ln(2)}{k}$ where k is the rate constant.

9. A reaction is second order with respect to reactant A. If the initial concentration of A is 0.1 M and the rate constant is 0.05 M^{-1} s^{-1}, calculate the concentration of reactant A after 20 seconds. Use the second-order integrated rate law: $[A]_t = \frac{1}{[A]_0 + kt}$ where $[A_0]$ is the initial

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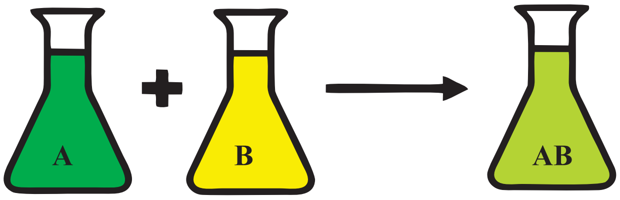
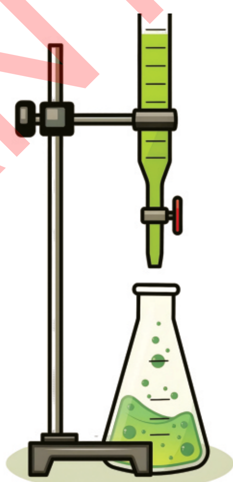
concentration, k is the rate constant and t is the time taken.



10. A certain amount of gas is initially in a container with a volume of 3 m^3 at a pressure of 1 atm. If the gas is compressed, and the volume is reduced to 2 m^3, what will be the new pressure of the gas if the temperature remains constant? Use the Boyle's law: $P_1V_1 = P_2V_2$ where P_1 is the initial pressure, V_1 is the initial volume, P_2 is the final pressure and V_2 is the final volume.

11. Calculate the pH value of a 0.05 M solution of HNO_3. Hint: Use the equation $pH = -log\left[H^+\right]$ where $\left[H^+\right]$ represents the concentration of H^+ ions.

12. You are given a solution of hydrochloric acid (HCl) with a concentration of 0.5 moles per litre (M). You need to prepare 250 milli-litres of a new solution with a concentration of 0.2 M. Determine the volume of the original 0.5 M solution needed and the amount of water (in milli-litres) to be added to achieve this new concentration. Use the dilution formula $M_1V_1 = M_2V_2$ where M_1 is the initial concentration, V_1 is the initial volume, M_2 is the final concentration and V_2 is the final volume.



Review Exercise 1

Each of the questions or incomplete statement below is followed by four suggested answers or completions. In each case, select the one that is the best of the choices.

(i) If $x^2 = -9$ then x = __________

(a) 3 (b) –3
(c) $3i$ (d) $\pm 3i$

(ii) The real part of complex number $z = 7i$ is:

(a) 0 (b) 7
(c) –7 (d) 1

(iii) The imaginary part of complex number $z = 8 + 10i$:

(a) 0 (b) 10
(c) 20 (d) 8

(iv) The additive inverse of $3+\frac{1}{2}i$ is ________

(a) $\frac{2}{6+i}$ (b) $\frac{2}{6-i}$
(c) $-3-\frac{1}{2}i$ (d) $3-\frac{1}{2}i$

(v) The multiplicative identity of complex number is:

(a) 0 (b) 1
(c) 2 (d) 3

(vi) The additive identity of a complex number is:

(a) 0 (b) 1
(c) 2 (d) 3

(vii) $\frac{\sqrt{-1250}}{\sqrt{2}}$ ________

(a) $-25i$ (b) 25
(c) –25 (d) $25i$

(viii) i^{10} = _______

(a) 1 (b) –1
(c) i (d) $-i$

(ix) The conjugate of $7 + 4i$ is ________

(a) $-7 + 4i$ (b) $7 - 4i$

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(c) $-7-4i$ (d) $7+4i$

(x) If we replace i by $-i$ in $z = x + iy$ then another complex number obtained is known as:

(a) Primer factor of z (b) Reciprocal of z
(c) Additive inverse of z (d) Complex conjugate of z

(xi) If $z_1 = 3 + i$ and $z_2 = 1 + 4i$ then $Re(z_1 - z_2) =$ _____

(a) -3 (b) 2
(c) 3 (d) 4

(xii) $|z_1 + z_2| =$ _______

(a) $|z_1| - |z_2|$ (b) $|z_1| - |\overline{z}_2|$
(c) $\overline{|z_1| + |z_2|}$ (d) $|z_1| + |z_2|$

(xiii) $x^2 + y^2 =$ _______

(a) $(x + yi)(x - yi)$ (b) $(x + y)(x - y)$
(c) $(x + yi)(x - y)$ (d) $(x + y)(x - yi)$

(xiv) If $|z^2| + 1 = |z^2 - 1|$ then z lies on:

(a) a circle (b) real axis
(c) imaginary axis (d) non of the above

(xv) the conjugate of the complex number six $x - i\cos 2x$ is:

(a) $\sin x + i\cos 2x$ (b) $\cos x - i\sin 2x$
(c) $-\sin x - i\cos 2x$ (d) $-\sin + i\cos 2x$

1. Why is the imaginary unit i defined as the square root of -1, and how does this help in solving equations that have no real solutions?
2. How do complex numbers help in finding the roots of equations that real numbers alone cannot solve?
3. Create a complex number whose real part is 3 and imaginary part is -4. What is its modulus?
4. Form a complex number z where the real part is half of the imaginary part and the modulus of z is 5.
5. For $z = 4 + 3i$, find the complex conjugate and the modulus of both z and its conjugate.
6. Explain why the modulus of a complex number is always a non-negative value.
7. If $z_1 = 2 + i$ and $z_2 = 3 - 2i$, calculate $z_1 + z_2$ and $z_1 - z_2$.
8. Show that multiplication of complex numbers is commutative by calculating $z_1 \cdot z_2$ and $z_2 \cdot z_1$ for $z_1 = 1 + 2i$ and $z_2 = -2 + i$.
9. Prove that the product of a complex number $z = 1 + 3i$ and its conjugate is a real number.
10. If $z = 2 - 5i$, calculate $z.\overline{z}$ and interpret the result.
11. Plot the complex number $z = 2 + 3i$ on an Argand diagram and then plot its conjugate. Describe what you observe.
12. Draw the complex numbers $z_1 = 1 + i$ and $z_2 = -1 - i$ on an Argand diagram and discuss their positions in relation to each other.
13. If z is a complex number such that $z + \overline{z} = 6$ and the imaginary part of z is 4, find z.
14. Solve for z in the equation $2z - 3\overline{z} = 4 + 6i$.
15. Describe a simple scenario where complex numbers might be used in everyday life.

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Summary

1. **Understanding Complex Numbers**: A complex number is expressed as $z = a + bi$, where a and b are real numbers, and i is the imaginary unit $(i^2 = -1)$
2. **Complex Conjugates**: The complex conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$.
3. **Modulus of Complex Numbers**: The modulus (or absolute value) of a complex number $|z| = \sqrt{a^2 + b^2}$, representing its distance from the origin in the complex plane.
4. **Properties of Complex Numbers**: Complex numbers follow algebraic properties like distributive, associative, and commutative laws, similar to real numbers.
5. **Properties of Conjugate**: The conjugate of a complex number has properties like $|z| = |-z| = |\bar{z}| = |-\bar{z}|$, $\bar{\bar{z}} = z$, $z\bar{z} = |z|^2$, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} + \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0.$
6. **Real and Imaginary Parts of Powers of Complex Numbers**: For a complex number $z = x + iy$, powers like z^n can be expanded to separate real and imaginary parts
7. **Solving Complex Linear Equations**: Solve complex coefficient linear equations with algebraic methods: substitution and elimination.
8. **Geometric Interpretation**: Complex numbers on the Argand diagram enable geometric interpretation of operations, with real parts on the x-axis and imaginary parts on the y-axis

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CHAPTER

2 Matrices

Matrices play an indispensable role in the world of digital imaging and graphics, a crucial element of modern digital communication and entertainment. These structured arrays of numbers are not just mathematical constructs; they are essential for transforming, compressing and rendering images and videos. By applying matrices, engineers and computer scientists can manipulate pixels in images to adjust brightness, contrast and even perform complex operations like rotation and scaling with precision and efficiency. This process is fundamental in everything from the crystal-clear visuals in blockbuster movies to the sharp and responsive graphics in video games. Matrices enable the seamless digital reconstruction of visual reality, making them a pivotal tool in the creation and compression of digital media. Their application in this field underscores the profound impact of mathematical concepts on enhancing and innovating visual technology.

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Students' Learning Outcomes

1. Display information in the form of matrix of order 2.
2. Solve situations involving sum, difference, and product of two matrices
3. Calculate the product of the scalar quantity and a matrix .
4. Evaluate the determinant and inverse of a matrix of order 2×2.
5. Solve the simultaneous linear equations in two variables using matrix inversion method and Cramer's rule
6. Explain, with examples, how mathematics plays a key role in the development of new scientific theories and technologies. [e.g., Mathematical models and simulations are used to design and optimize new materials and drugs, and to understand the behaviour of complex systems such as the human brain.]
7. Apply concepts of matrices to real world problems (such as engineering, economics, computer graphics, and physics).

Knowledge

- **Understanding of a 2×2 Matrix**: Comprehension of the structure and representation of a 2×2 matrix.
- **Matrix Operations**: Familiarity with the rules and principles for performing addition, subtraction, and multiplication of matrices.
- **Scalar Multiplication in Matrices**: Knowledge of how scalar values interact with matrices during multiplication.
- **Determinant and Inverse of 2×2 Matrices**: Understanding the concepts and calculation methods for determinants and inverses of 2×2 matrices.
- **Matrix Inversion and Cramer's Rule**: Comprehension of matrix inversion and Cramer's rule as techniques for solving systems of linear equations.
- **Role of Mathematics in Science and Technology**: Awareness of how mathematical models and simulations are integral to scientific and technological advancements.
- **Application of Matrices in Various Fields**: Understanding of the practical applications of matrices in engineering, economics, computer graphics, and physics.

Skill

- **Matrix Representation**: Ability to organize and present data or information in a 2×2 matrix format.
- **Performing Matrix Operations**: Competence in executing matrix addition, subtraction, and multiplication, especially with 2x2 matrices.
- **Scalar and Matrix Multiplication**: Skill in multiplying matrices by scalar quantities and understanding the outcomes.
- **Calculating Determinants and Inverses**: Proficiency in determining the determinant and inverse of 2×2 matrices and interpreting their significance.
- **Solving Linear Systems**: Capability to use matrix inversion and Cramer's rule to find solutions to systems of linear equations.
- **Explaining Mathematical Applications**: Ability to articulate and exemplify the role of mathematics in the development of new scientific theories and technologies.
- **Practical Application of Matrix Concepts**: Aptitude for applying matrix theory to solve real-world problems in various professional and academic fields.

Pre & Post Requisite

Class 10
Chapter # 2
Matrices

Class 11
Chapter # 2
Matrices and Determinants

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Student Learning Outcomes

✧ Display information in the form of matrix of order 2 by 2

2.1 INTRODUCTION TO MATRICES

The word 'matrices' is plural of the word 'matrix'. The term matrix was first introduced by the mathematician Arther Cayley in 1860. The knowledge of matrices is necessary in various areas of Mathematics. It has widely been used in the fields of pure mathematics, statistics, engineering and physical and social sciences. Thus, matrix theory finds an important place in modern age and has become an integral part of mathematics.

Matrices make presentation of numbers clearer and make calculations easier.

The following table presents the sales figures for Smartphones and Tablets in January and February.

	January	February
Smartphones	300	350
Tablets	250	280

The information is readily available when presented in this way. For example, if we want to know the sales figures for Tablets in February, we go along the row 'Tablets' and column 'February' and find that it is 280. As long as we remember what each number represents, we could remove the row and column headings and write just the numbers, enclosing them in square brackets or parentheses such as

$$\begin{bmatrix} 300 & 350 \\ 250 & 280 \end{bmatrix}$$

For instance, if the temperatures for City A and City B over Day 1 and Day 2 are as follows:

- City A, Day 1: 75°F
- City A, Day 2: 80°F
- City B, Day 1: 68°F
- City B, Day 2: 72°F

	Day 1	Day 2
City A	75	80
City B	68	72

Let us denote this by A.

Thus $A = \begin{bmatrix} 75 & 80 \\ 68 & 72 \end{bmatrix}$ is a matrix.

A matrix is a rectangular array (arrangements) of real numbers enclosed in square brackets. Each number in a matrix is called an element or entry of the matrix.

For example $\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ and

$\begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & -2 \\ -4 & -3 & -3 \end{bmatrix}$ are all matrices. In the matrix

$\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$ the number $2,3,6,5$ are the elements or entries of the matrix. Similarly in the matrix

$\begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & -2 \\ -4 & -3 & -3 \end{bmatrix}$, the numbers

$3,4,2,-1,1,-2,-4,-3,-5$ are the elements or entries of the given matrix. Matrices are frequently denoted by capital letters such as A, B, C and so on. Thus we can write

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix},\ C = \begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & -2 \\ -4 & -3 & -3 \end{bmatrix}$$

2.1.2 Rows and Columns of Matrix

The rows of a matrix run horizontally, and the columns of a matrix run vertically.

To make the idea more clear, let us consider the following data and make a matrix .

In a survey of 600 customers regarding their preferences for three different types of electronic devices, the following information was obtained:

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- 200 male customers preferred smartphones
- 150 male customers preferred tablets
- 50 male customers preferred laptops
- 180 female customers preferred smartphones
- 90 female customers preferred tablets
- 30 female customers preferred laptops

We can arrange these data in a rectangular array as follows:

Gender	Smartphones	Tablets	Laptops
Males	200	150	50
Females	180	90	30

or as the matrix

$$\begin{bmatrix} 200 & 150 & 50 \\ 180 & 90 & 30 \end{bmatrix}$$

This matrix has two rows (representing males and females) and three columns (representing "smartphones," "tablets," and "laptops").

The matrix we developed in above Example has 2 rows and 3 columns. In general, a matrix with 'm' rows and 'n' columns is called an m by n matrix.The matrix we developed in previous example is a 2 by 3 matrix and contains $2\times3 = 6$ entries. An m by n matrix will contain $m \times n$ entries.

$$\begin{bmatrix} 200 & 150 & 50 \\ 180 & 90 & 30 \end{bmatrix}$$

2.1.3 Order or Dimension of a Matrix

A matrix with m rows and n columns has order $m\times n$ (read "m by n").

If a matrix has order $m\times n$ then m represents the number of rows and n represents the number of columns.

For example, consider

$$A=\begin{bmatrix} -3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \text{ and } B=\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$$

The order of the matrix A is 2-by-3 or A is a 2-by-3 matrix. Similarly the order of the matrix B is 2-by-2 or B is a 2-by-2 matrix.

Note : Order of a matrix $m\times n$ does not mean to multiply m *and* n.

Remember

Sometimes we will prefer to write the order of a matrix as m-by-n or sometimes $(m\times n)$.
Order of a matrix is also called dimension or size.

Example 2.1:

Write the number of rows and column of the following matrices and hence mention their orders.

Solution:

i) $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ **ii)** $B=\begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix}$

i) Given $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, Order$=2\times2$

ii) Given $B=\begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix}$ Order$=2\times3$

2.1.4 Equality of two Matrices

Two given A and B are said to be equal if:

(i) Both the matrices are of the same order i.e. they respectively have the same number of rows and columns.

(ii) The elements in the corresponding positions in A and B are equal.

For example, the matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1+1 & \frac{12}{4} \\ 4+2 & \frac{10}{2} & \frac{8}{2} \end{bmatrix} \text{ are equal}$$

whereas the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ are not equal.

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Skill 2.1

Understanding of a 2×2 Matrix: Comprehension of the structure and Representation of a 2×2 matrix.

Exercise 2.1

1. Using appropriate understanding of data, decide which data will be represented in rows and which in columns, and display the information in the form of a matrix as specified.

(i) A school's cafeteria tracks the number of vegetarian and non-vegetarian meals sold over two days. The data is as follows:

- Day 1: 120 vegetarian meals, 80 non-vegetarian meals
- Day 2: 150 vegetarian meals, 100 non-vegetarian meals

(ii) A company's quarterly revenue (in thousands of dollars) from two departments (Sales and Marketing) is recorded as follows:

- Quarter 1: Sales PKR 200k, Marketing PKR 150k
- Quarter 2: Sales PKR 220k, Marketing PKR 160k

(iii) An animal shelter tracks the number of dogs and cats adopted over two months. The data is as follows:

- Month 1: 30 dogs, 45 cats
- Month 2: 40 dogs, 50 cats

(iv) A class tracks the scores of two students in two subjects (History and Geography). The scores are as follows:

- Student A: History 85, Geography 90
- Student B: History 80, Geography 85

(v) A survey was conducted in two cities (City A and City B) regarding three different transportation methods (Bus, Train, Car). The number of people using each method is as follows:

- City A: 120 people use Bus, 80 people use Train, 200 people use Car
- City B: 150 people use Bus, 100 people use Train, 250 people use Car

(vi) A small company has recorded the sales (in units) of three products (Product X, Product Y, and Product Z) over four quarters. The sales data is as follows:

- Q1: 50 units of Product X, 70 units of Product Y, 30 units of Product Z
- Q2: 65 units of Product X, 80 units of Product Y, 45 units of Product Z
- Q3: 75 units of Product X, 60 units of Product Y, 50 units of Product Z
- Q4: 90 units of Product X, 85 units of Product Y, 55 units of Product Z

2. If $A = \begin{bmatrix} 3 & 2 & -4 \\ -2 & 5 & 0 \\ 2 & 1 & 5 \\ -3 & 4 & 6 \end{bmatrix}$, then write down the following elements.

(i) a_{12} **(ii)** a_{23} **(iii)** a_{32}

(iv) a_{43} **(v)** a_{13} **(vi)** a_{33}

3. List the order of the following matrices.

i) $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ **ii)** $B = \begin{bmatrix} -4 \end{bmatrix}$

iii) $C = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 5 \end{bmatrix}$ **iv)** $F = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & -1 \end{bmatrix}$

v) $E = \begin{bmatrix} 3 & 2 \end{bmatrix}$ **vi)** $D = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

4. Which of the following matrices are equal.

$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$,

$C = \begin{bmatrix} 1+1 & 3+2 \\ 4 & 2+1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 4+1 \\ 1 & 3 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 2 & -3 \\ u & o \end{bmatrix}$ and $B = \begin{bmatrix} v & -3 \\ 5 & w \end{bmatrix}$, for what

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values of u, v and w, A and B are equal ?

6. If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$,

find the values of a,b,c,x,y and z.

7. Solve the following matrix for a,b,c,d.

$$\begin{bmatrix} a+b & b+2c \\ 2c+d & 2a-d \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 8 & 0 \end{bmatrix}$$

2.2 TYPES OF MATRICES

Matrices are essential mathematical structures used to organize and manipulate data. Understanding the different types of matrices is crucial for effectively applying matrix operations and solving complex problems. Each type of matrix serves a unique purpose, allowing for greater flexibility and efficiency in mathematical computations.

This section explores various types of matrices, each defined by specific characteristics and properties. These types include:

2.2.1 Square-matrix

A matrix in which number of rows and columns are equal is called a square matrix. For example, the matrix $\begin{bmatrix} l & m \\ n & p \end{bmatrix}$ has two rows and two columns, so it is a square matrix. The order of the matrix $\begin{bmatrix} l & m \\ n & p \end{bmatrix}$ is 2-by-2 but for brevity we write the above matrix as a square matrix of order 2 or 2-square matrix. Similarly the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ -1 & -4 & 2 \end{bmatrix}$ is a square matrix of order 3 or 3-square matrix. As a special case the matrix consisting of a single element say [3] is also a square matrix of order 1 or a 1-square matrix.

2.2.2 Rectangular-matrix

A matrix whose number of rows and number of columns are not equal is called rectangular matrix. Thus, the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ is a rectangular matrix because its number of rows are $2 = m$ and its number of columns are $3 = n$; obviously $2 \neq 3$. Similarly, the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 5 & -3 \end{bmatrix}$ is also a rectangular matrix with order 3-by-2, i.e., number of rows and number of columns are not equal.

2.2.3 Row-matrix

A matrix which has only row in it is called a row-matrix.

For example, the matrices

$[a, b], [1 \quad 3 \quad 4], [2 \quad 4 \quad 6 \quad 8]$ are all row-matrices.

2.2.4 Column-matrix

A matrix which has only one column in it is called a column matrix.

For example, $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ are all column matrices.

2.2.5 Zero matrix or Null matrix

Any matrix (whether it is rectangular or square) in which all the elements (entries) are equal to zero is said to be a Zero matrix or a Null matrix. Thus,

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are some examples of zero matrices or null matrices.

The following points are very important to remember.

i) A zero or null matrix is not necessarily a square matrix.

ii) A null matrix is generally donated by O.

iii) Sometimes we prefer to put the order of a null matrix as a proper subscript with the general symbol O. Thus, $O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null-matrix of order 2.

$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null-matrix of order 2-by-3.

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Note: In matrix manipulation, the Zero matrix is similar to zero in regular algebra. When compatible for multiplication (discussed later), any matrix multiplied by the Zero matrix becomes a zero matrix. Choose the appropriate zero matrix order for problem-solving.

2.2.6 Diagonal matrix

A square matrix in which all elements are zero except the diagonal elements is known as diagonal matrix. For example,

$A=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$ is a diagonal matrix of order 2 and

$B=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -5\end{bmatrix}$ is a diagonal matrix of order 3.

2.2.7 Scalar matrix

A square matrix in which all the elements lying on the main diagonal of the matrix are equal and the remaining elements of the main diagonal are all zero is called scalar matrix.

Equivalently scalar matrix can also be defined as,

A diagonal matrix in which all the diagonal elements are equal is said to be a scalar matrix.

For instance, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}7 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 7\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

Scalar matrix is a special case of a diagonal matrix.

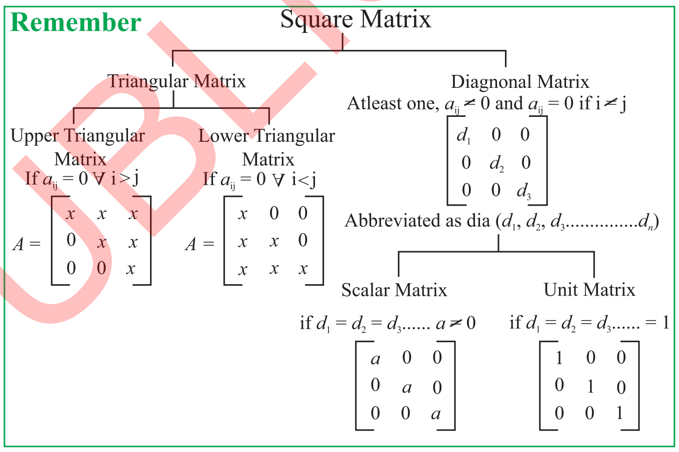
Note
Every scalar matrix is a diagonal matrix but every diagonal matrix is not necessarily a scalar matrix. For example,

$\begin{bmatrix}4 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -1\end{bmatrix}$ is a diagonal matrix, because all elements of the matrix are zero except the elements of the main diagonal but the above matrix is not a scalar matrix because elements lying on the main diagonal are not equal.

2.2.8 Identity matrix

The identity matrix (also called the unit matrix) is a square matrix and is denoted by I. It is characterized by the fact that all elements on its main diagonal are 1's whereas all other elements are zero. Sometimes it is useful to write the order of the identity matrix as a subscript of I. Thus, $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is the identity matrix of order 2 and $I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is the identity matrix of order 3.)

When there is no danger of ambiguity the identity matrices I_2 and I_3 are simply denoted by I.



2.2.9 Transpose of a matrix

Suppose we have a matrix A of any given order. The matrix obtained by interchanging mutually the rows and columns in A is called the transpose of A and is denoted by A^t. For example

If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ then $A^t=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

2.2.10 Symmetric matrix

A square matrix A is said to be symmetric if the transpose of A denoted by A^t is again equal to A, i.e., $A = A^t$. For example,

If $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ then $A^t=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$,

We see that $A = A^t$, so $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ is symmetric.

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Similarly if $B=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$ then $B'=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}=B$

i.e., $B=B^t$, so $B=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$ is a symmetric matrix.

2.2.11 Skew-Symmetric matrix

A given square matrix A is said to be Skew-Symmetric if $A^t=-A$.

For example: If $A=\begin{bmatrix}0&-3\\3&0\end{bmatrix}$ and $A^t=\begin{bmatrix}0&3\\-3&0\end{bmatrix}$

then,

$A^t=\begin{bmatrix}0&3\\-3&0\end{bmatrix}=-\begin{bmatrix}0&-3\\3&0\end{bmatrix}=-A$ i.e. $A^t=-A$ so

$A=\begin{bmatrix}0&-3\\3&0\end{bmatrix}$ is a skew symmetric matrix.

Exercise 2.2

1. Which of the following are square and which are rectangular matrices?

i) $A=\begin{bmatrix}2&3\\0&5\end{bmatrix}$ **ii)** $B=\begin{bmatrix}6&3&-1\\1&5&2\end{bmatrix}$

iii) $C=\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$ **iv)** $B=\begin{bmatrix}-5\end{bmatrix}$

v) $E=\begin{bmatrix}-3&4\end{bmatrix}$ **vi)** $F=\begin{bmatrix}-1\\7\end{bmatrix}$

2. Write transpose of the following matrices:

i. $P=\begin{bmatrix}1&2\\3&1\end{bmatrix}$ ii. $Q=\begin{bmatrix}l&m\\n&p\end{bmatrix}$

iii. $R=\begin{bmatrix}6\end{bmatrix}$ iv. $S=\begin{bmatrix}-5&1\\-2&1\\4&4\end{bmatrix}$

v. $T=\begin{bmatrix}6&7&8\\13&1&3\\2&4&5\end{bmatrix}$

3. Which of the following matrices are transpose of each other?

i. $A=\begin{bmatrix}a_1&a_2\\b_1&b_2\end{bmatrix}$ ii. $B=\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}$

iii. $C=\begin{bmatrix}-3&1&-1\\4&2&7\end{bmatrix}$ iv. $D=\begin{bmatrix}-3&4\\1&2\\-1&7\end{bmatrix}$

4. Which of the following matrices are symmetric?

i. $A=\begin{bmatrix}5&-7\\-1&5\end{bmatrix}$ ii. $B=\begin{bmatrix}-1&2\\2&3\end{bmatrix}$

iii. $C=\begin{bmatrix}3&4\\5&6\end{bmatrix}$ iv. $D=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$

5. Which of the following matrices are skew-symmetric?

i. $A=\begin{bmatrix}0&4\\-4&0\end{bmatrix}$ ii. $B=\begin{bmatrix}0&-5\\5&0\end{bmatrix}$

iii. $C=\begin{bmatrix}0&7\\7&0\end{bmatrix}$ iv. $D=\begin{bmatrix}0&3&2\\-3&0&1\\-2&-1&0\end{bmatrix}$

Student Learning Outcomes

- Solve situations involving sum, difference, and product of two matrices.
- Calculate the product of the scalar quantity and a matrix.

2.3 ADDITION AND SUBTRACTION OF MATRICES

This section covers the basics of adding and subtracting matrices, essential for solving matrix equations and analyzing data in various fields.

2.3.1 Conformability for Addition/subtraction of matrices

Suppose that we have two matrices. These two matrices may be added or subtracted if andonly if they are of the same order.

Example 2.2:

i) Let $A=\begin{bmatrix}1&3\\-3&2\end{bmatrix}$ and $B=\begin{bmatrix}4&7\\10&13\end{bmatrix}$ then these

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matrices are conformable for addition and subtraction because both are 2-by-2 matrices,that is both matrices are of the same order.

ii) Let $C=\begin{bmatrix}7 & 3 & 1\\ -2 & 0 & 5\end{bmatrix}$ and $D=\begin{bmatrix}9 & 5 & 13\\ 10 & -1 & 1\\ 2 & 0 & 3\end{bmatrix}$, then

C and D are not conformable for addition and subtraction because their orders are not same.

2.3.2 Addition and subtraction of matrices

a) Addition of two matrices

Suppose A and B are conformable for addition. Their sum $A+B$ is obtained by adding corresponding elements of the matrices A and B.

Let $A=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}$

We see that A and B both are 2-by-2 matrices, so these are conformable for addition.

$$A+B=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}+\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}=\begin{bmatrix}3+4 & 8+0\\ 4+1 & 6-9\end{bmatrix}=\begin{bmatrix}7 & 8\\ 5 & -3\end{bmatrix}$$

b) Subtraction of two matrices

Let A and B are two matrices which are of the same order and so these are conformable for subtraction. Their difference $A–B$ is obtained by subtracting each element of B form the corresponding element of A. For example

$$Let\ A=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}\ and\ B=\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}$$

Both the matrices are of the same order, so these are conformable for subtraction.

$$A-B=\begin{bmatrix}3 & 8\\ 4 & 6\end{bmatrix}-\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}$$

$$=\begin{bmatrix}3-4 & 8-0\\ 4-1 & 6-(-9)\end{bmatrix}=\begin{bmatrix}3-4 & 8-0\\ 4-1 & 6+9)\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 8\\ 3 & 15\end{bmatrix}$$

2.3.3 Multiplication of a matrix by a real number

Let A be any matrix and k is any real number. The matrix obtained by multiplying each element of A by the real number k is called the scalar multiplication of A by k and it is denoted by kA.

For example : $2\begin{bmatrix}4 & 0\\ 1 & -9\end{bmatrix}=\begin{bmatrix}8 & 2\\ 2 & -18\end{bmatrix}$

i. If $A=\begin{bmatrix}6 & 2\\ -3 & 1\end{bmatrix}$

then $3A=\begin{bmatrix}6\times 3 & 2\times 3\\ -3\times 3 & 1\times 3\end{bmatrix}=\begin{bmatrix}18 & 6\\ -9 & 3\end{bmatrix}$

ii. If $B=\begin{bmatrix}5 & 4 & 7\\ -3 & a & b\end{bmatrix}$

then $7B=\begin{bmatrix}7\times 5 & 7\times 4 & 7\times 7\\ 7\times(-3) & 7a & 7b\end{bmatrix}$

$=\begin{bmatrix}35 & 28 & 49\\ -21 & 7a & 7b\end{bmatrix}$

2.3.4 Commutative and associative laws under addition

a) Commutative law under addition

Let A and B are two given matrices of the same order then $A+B=B+A$. This is called commutative law of matrices under addition.

Now we give some examples in support of the above law.

Example 2.3:

Let $A=\begin{bmatrix}2 & 5\\ 4 & 7\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 1\\ -3 & 6\end{bmatrix}$, then A & B are of the same order and hence these matrices can be added.

Solution:

$$A+B=\begin{bmatrix}2 & 5\\ 4 & 7\end{bmatrix}+\begin{bmatrix}-2 & 1\\ -3 & 6\end{bmatrix}=\begin{bmatrix}2+(-2) & 5+1\\ 4+(-3) & 7+6\end{bmatrix}=\begin{bmatrix}0 & 6\\ 1 & 13\end{bmatrix}$$

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Again $B+A=\begin{bmatrix}-2 & 1\\ -3 & 6\end{bmatrix}+\begin{bmatrix}2 & 5\\ 4 & 7\end{bmatrix}$

$=\begin{bmatrix}-2+2 & 1+5\\ -3+4 & 6+7\end{bmatrix}=\begin{bmatrix}0 & 6\\ 1 & 13\end{bmatrix}$

i.e. $B+A=\begin{bmatrix}0 & 6\\ 1 & 13\end{bmatrix}$

This proves that $A+B=B+A$

(b) Associative law for matrix addition

Let A, B and C are three matrices of the same order, that is, these are conformable for addition.

Then $A+(B+C)=(A+B)+C$

This is called associative law for matrix addition.

Now we give an example in support of the above law.

Example 2.4:

Let $A=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}, B=\begin{bmatrix}4 & -5\\ 6 & 7\end{bmatrix}$ and $C=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$,

then all these matrices are of the same order, so these are conformable for addition.

Now

$$A+(B+C)=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\left(\begin{bmatrix}4 & -5\\ 6 & 7\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\begin{bmatrix}4+3 & -5-2\\ 6+1 & 7+0\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\begin{bmatrix}7 & -7\\ 7 & 7\end{bmatrix}$$

$$=\begin{bmatrix}-1+7 & 2-7\\ 4+7 & -3+7\end{bmatrix} \therefore A+(B+C)=\begin{bmatrix}6 & -5\\ 11 & 4\end{bmatrix}$$

Again

$$(A+B)+C=\left(\begin{bmatrix}-1 & 2\\ 4 & -3\end{bmatrix}+\begin{bmatrix}4 & -5\\ 6 & 7\end{bmatrix}\right)+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-1+4 & 2+(-5)\\ 4+6 & -3+7\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}3 & -3\\ 10 & 4\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}3+3 & -3-2\\ 10+1 & 4+0\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5\\ 11 & 4\end{bmatrix}$$

$\therefore A+(B+C)=(A+B)+C$

2.3.5 Additive identity of matrices

In ordinary arithmetic 0 (zero) is called additive identity because when 0 is added to any real number or when any real number is added to 0 , the value of the number does not change. For example,

$0+5=5+0=5,$

$\frac{1}{3}+0=0+\frac{1}{3}=\frac{1}{3}$

and $\sqrt{7}+0=0+\sqrt{7}=\sqrt{7}$ and so on.

In theory of matrices, zero matrix (or Null-matrix) denoted by O performs the same function as 0 in ordinary arithmetic because when O is added to any matrix A or if A is added to the matrix O, the matrix A does not change provided O and A are conformable for addition. In such a situation $A + O = O + A = A$.

Definition:

In the theory of matrix a Zero or (Null) matrix of some specific order serves as the additive identity for the matrices of the same specific order.

Here,we give some examples to make the idea more clear.

Example 2.5:

If $A=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}$ then the additive identity for such a case is $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ because A and O are conformable for addition.

$$A+O=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}3+0 & -1+0\\ 2+0 & 5+0\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}=A$$

Also,

$$O+A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}=\begin{bmatrix}0+3 & 0-1\\ 0+2 & 0+5\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}=A$$

Therefore, $A+O=O+A=A$.

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$O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is additive identity of $A=\begin{bmatrix}3&-1\\2&5\end{bmatrix}$. But at the same time $O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is additive identity for all 2-square matrices.

2.3.6 Additive inverse of a matrix

Let A and B be two matrices of the same order. If A and B are so related together that $A+B=O=B+A$ where O is the identity matrix of the same order as that of order of A or B, then B is called the additive inverse of A and likewise A is called the additive inverse of B. In other words $B=-A$ which implies that $A=-B$ i.e, A and B are additive inverses of each other.

Example 2.6:

Prove that $P=\begin{bmatrix}3&2&-1\\-2&4&6\end{bmatrix}, Q=\begin{bmatrix}-3&-2&1\\2&-4&-6\end{bmatrix}$ are additive inverse of each other.

Solution:

Since,

$$P+Q=\begin{bmatrix}3&2&-1\\-2&4&6\end{bmatrix}+\begin{bmatrix}-3&-2&1\\2&-4&-6\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}=O$$

$$Q+P=\begin{bmatrix}-3&-2&1\\2&-4&-6\end{bmatrix}+\begin{bmatrix}3&2&-1\\-2&4&6\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}=O$$

Hence, P and Q are additive inverses of each other.

Skill 2.3

- **Matrix Operations:** Familiarity with the rules and principles for performing addition, subtraction
- **Scalar Multiplication in Matrices:** Knowledge of how scalar values interact with matrices during multiplication.

Exercise 2.3

1. Let A and B be 2-by-3 matrices and let C and D be 2-square matrices. Which of the following matrix operations are defined. For those which are defined, give the dimension of the resulting matrix.

   i) $A+B$ ii) $B+D$
   iii) $3A-2C$ iv) $7C+2D$

2. Multiply the following matrices by the real numbers as indicated.

   i) Multiply $A=\begin{bmatrix}1\\2\\3\end{bmatrix}$ by 2

   ii) Multiply $B=\begin{bmatrix}a&c\\d&f\end{bmatrix}$ by $p\in R$

3. If $A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$ and $B=\begin{bmatrix}-3&-2\\1&-5\\4&3\end{bmatrix}$, then find $3A-B$.

6. If $A=\begin{bmatrix}2&-2\\4&2\\-5&1\end{bmatrix}$ and $B=\begin{bmatrix}8&0\\4&-2\\3&6\end{bmatrix}$, then find the matrix X such that $2A+3X=5B$.

7. Find x, y, z and w if

   $$3\begin{bmatrix}x&y\\z&w\end{bmatrix}=\begin{bmatrix}x&6\\-1&2w\end{bmatrix}+\begin{bmatrix}4&x+y\\3+w&3\end{bmatrix}.$$

8. Find X and Y if $X+Y=\begin{bmatrix}5&2\\0&9\end{bmatrix}$ and $X-Y=\begin{bmatrix}3&6\\0&-1\end{bmatrix}$.

9. Let $A=\begin{bmatrix}2&-3\\4&5\end{bmatrix}, B=\begin{bmatrix}2&5\\-1&3\end{bmatrix}$ and $C=\begin{bmatrix}3&-1\\0&4\end{bmatrix}$.

   If $c=2$ and $d=-4$ then verify that:

   i) $(c+d)A=cA+dA$

   ii) $c(A+B)=cA+cB$

   iii) $cd(A)=c(dA)$

10. Let $A=\begin{bmatrix}-1&2&3\\4&2&0\\-3&2&5\end{bmatrix}, B=\begin{bmatrix}3&-1&2\\-5&3&4\\-3&-4&0\end{bmatrix}$ and $C=\begin{bmatrix}2&-3&6\\0&4&-1\\-5&1&3\end{bmatrix}$.

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Compute the following if possible.

i) $A+2B$ **ii)** $3A-4B$

iii) $(A+B)-C$ **iv)** $A+(B+C)$

11. Prove that in the following matrices commutative law of addition holds.

i) $A=\begin{bmatrix}7 & 1\\ 2 & 4\end{bmatrix}, B=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$

ii) $C=\begin{bmatrix}-3 & 4 & -5\\ 2 & 3 & 1\end{bmatrix}, D=\begin{bmatrix}-3 & -4 & 5\\ 1 & 2 & 3\end{bmatrix}$

12. Find the additive inverse of the following matrices.

i) $A=\begin{bmatrix}3 & 4\\ 6 & 2\end{bmatrix}$

ii) $B=\begin{bmatrix}a & -a & b\\ -c & a & -b\\ l & m & n\end{bmatrix}$

13. Show that the following matrices are additive inverses of each other.

i) $A=\begin{bmatrix}1 & -2 & 3\end{bmatrix}, B=\begin{bmatrix}-1 & 2 & -3\end{bmatrix}$

ii) $C=\begin{bmatrix}a & -b\\ -c & d\end{bmatrix}, D=\begin{bmatrix}-a & b\\ c & -d\end{bmatrix}$

iii) $E=\begin{bmatrix}1 & -2 & -4\\ 2 & 1 & 3\\ -3 & 4 & -2\end{bmatrix}, F=\begin{bmatrix}-1 & 2 & 4\\ -2 & -1 & -3\\ 3 & -4 & 2\end{bmatrix}$

2.4 MULTIPLICATION OF MATRICES

2.4.1 Conformability for multiplication of matrices

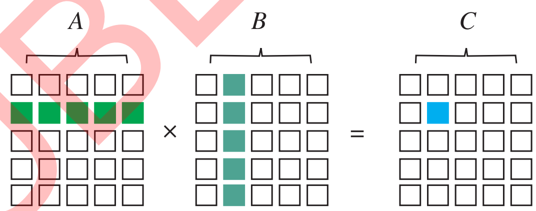
Two matrices A and B are said to be conformable for multiplication AB, only when the number of columns of matrix A is equal to the number of rows of matrix B. The product AB, which is not the same as the product BA, is conformable for multiplication only if A is an $m \times p$ and matrix B is a $p \times n$ matrix. The product AB will then be an $m \times n$ matrix. A convenient way to determine whether two matrices are conformable for multiplication, and to determine the order of the resultant matrix, is to write the orders of the two matrices side-by-side as shown below.

This shows that A and B are conformable for multiplication as $p = p$. It also indicates that the order of the product AB is $m \times n$.

Thus $A_{m\times p} \times B_{p\times n} = AB_{m\times n}$.

Here, A and B are not conformable for multiplication since $n \neq m$. Thus,product BA is not defined.

For matrix multiplication, the operation is row by column. Thus, to obtain the product AB, we multiply each element of a row of A by the corresponding element of a column of B and then we add these products.



For Example:

Suppose A is a 3-by-4 matrix, B is a 4-by-2 matrix and C is a 4-by-3 matrix.

Then AB is defined and it is 3-by-2 matrix. AC is defined and it is 3-by-3 i.e. 3 -square matrix CA is also defined and it is a 4-by-4 matrix while BA, CB and BA are undefined.

Example 2.7:

If $A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$, then

i) is it possible to find both AB and BA ?

ii) find the product if possible.

Solution:

Given $A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$

i) First we check whether AB is possible.

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Number of columns in A are 2 and Number of rows in B are 2. Therefore, AB is possible.

$$AB=\begin{bmatrix}2&3\\1&4\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}2\times3+3\times5\\1\times3+4\times5\end{bmatrix}=\begin{bmatrix}6+15\\3+20\end{bmatrix}=\begin{bmatrix}21\\23\end{bmatrix}$$

$$\therefore\ AB=\begin{bmatrix}21\\23\end{bmatrix}$$

Now, we discuss the existence of the product BA.

Since number of columns in $B=1$ and numbers of rows in $A=2$. Since $1\neq2$, so BA is not possible.

ii) The only possible product is AB and its value is

$$AB=\begin{bmatrix}21\\23\end{bmatrix}$$

2.4.2 Commutative Law of Multiplication of Matrices

Commutative law of multiplication of matrices in general does not hold as shown in the following example.

Example 2.8:

Solution:

Let $A=\begin{bmatrix}6&3\\2&5\end{bmatrix}$, $B=\begin{bmatrix}-3&2\\1&5\end{bmatrix}$.

Determine whether $AB=BA$.

$$AB=\begin{bmatrix}6&3\\2&5\end{bmatrix}\begin{bmatrix}-3&2\\1&5\end{bmatrix}=\begin{bmatrix}-15&27\\-1&29\end{bmatrix}$$

$$BA=\begin{bmatrix}-3&2\\1&5\end{bmatrix}\begin{bmatrix}6&3\\2&5\end{bmatrix}=\begin{bmatrix}-14&1\\16&28\end{bmatrix}$$

Here, we have $AB\neq BA$.

The above example shows that matrix multiplication is not commutative in general, that is $AB\neq BA$.

Though it can happen that $AB=BA$.

Now we give an example of matrices for which commutative property of multiplication is true.

Example 2.9: If Let $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}2&2\\3&5\end{bmatrix}$, show that $AB=BA$.

Solution:

$$AB=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2&2\\3&5\end{bmatrix}=\begin{bmatrix}1\times2+2\times3&1\times2+2\times5\\3\times2+4\times3&3\times2+4\times5\end{bmatrix}$$

$$=\begin{bmatrix}2+6&2+10\\6+12&6+20\end{bmatrix}=\begin{bmatrix}8&12\\18&23\end{bmatrix}$$

Now

$$BA=\begin{bmatrix}2&2\\3&5\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}2\times1+2\times3&2\times2+2\times4\\3\times1+5\times3&3\times2+5\times4\end{bmatrix}$$

$$=\begin{bmatrix}2+6&4+8\\3+15&6+20\end{bmatrix}=\begin{bmatrix}8&12\\18&26\end{bmatrix}$$

Here, we have $AB=BA$. So the given matrices commute.

2.4.3 Associative law under multiplication

If A, B and C are three matrices such that A is m-by-n matrix, B is n-by-p matrix and C is p-by-q matrix. Then $A(BC)=(AB)C$

This is called the associative law of matrices under multiplication.

Example 2.10:

If $A=\begin{bmatrix}1\\2\end{bmatrix}$, $B=\begin{bmatrix}3&2\end{bmatrix}$, $C=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, then verify that $A(BC)=(AB)C$.

Solution:

$$BC=\begin{bmatrix}3&2\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}3\times1+2\times31&3\times2+2\times4\end{bmatrix}$$

$$=\begin{bmatrix}3+6&6+8\end{bmatrix}=\begin{bmatrix}9&14\end{bmatrix}$$

$$A(BC)=\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}9&14\end{bmatrix}=\begin{bmatrix}1\times9&1\times14\\2\times9&2\times4\end{bmatrix}$$

$$=\begin{bmatrix}9&14\\18&28\end{bmatrix}\ \ldots\ldots\ (\text{i})$$

$$AB=\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3&2\end{bmatrix}=\begin{bmatrix}1\times3&1\times2\\2\times3&2\times2\end{bmatrix}=\begin{bmatrix}3&2\\6&4\end{bmatrix}$$

$$(AB)C=\begin{bmatrix}3&2\\6&4\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}3\times1+2\times3&3\times2+2\times4\\6\times1+4\times3&6\times2+4\times4\end{bmatrix}$$

$$=\begin{bmatrix}3+6&6+8\\6+12&12+6\end{bmatrix}=\begin{bmatrix}9&14\\18&28\end{bmatrix}\ \ldots\ldots\ (\text{ii})$$

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From (i) and (ii), we have $A(BC)=(AB)C$.

This verifies associative law of matrix multiplication.

2.4.4 Distributive laws of multiplication over addition

Let A, B and C are three matrices such that A is a m-n by-n matrix, B is a n-by-p matrix and C is also a n-by-p matrix.

$A(B+C)=AB+AC$ and $(A+B)C=AC+BC$

These are called distributive laws of multiplication over addition.

We try to verify these laws by choosing matrices of suitable order in the following example.

Example 2.11:

If $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, B=\begin{bmatrix}5 & 3\\2 & 2\end{bmatrix}, C=\begin{bmatrix}6 & 2\\5 & 1\end{bmatrix}$. Verify the distributive of multiplication over addition.

Solution:

We see A, B and C are 2-by-2 matrices. Hence they are conformable for multiplication and addition.

$$B+C=\begin{bmatrix}5 & 3\\2 & 4\end{bmatrix}+\begin{bmatrix}6 & 2\\5 & 1\end{bmatrix}=\begin{bmatrix}5+6 & 3+2\\2+5 & 4+1\end{bmatrix}=\begin{bmatrix}11 & 5\\7 & 5\end{bmatrix}$$

$$A(B+C)=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}11 & 5\\7 & 5\end{bmatrix}=\begin{bmatrix}1\times 11+2\times 7 & 1\times 5+2\times 5\\3\times 11+4\times 7 & 3\times 5+4\times 5\end{bmatrix}$$

$$=\begin{bmatrix}11+14 & 5+10\\33+28 & 15+20\end{bmatrix}=\begin{bmatrix}25 & 15\\61 & 35\end{bmatrix}\ldots\ldots\text{(i)}$$

$$AB=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 3\\2 & 4\end{bmatrix}=\begin{bmatrix}1\times 5+2\times 2 & 1\times 3+2\times 4\\3\times 5+4\times 2 & 3\times 3+4\times 4\end{bmatrix}$$

$$=\begin{bmatrix}5+4 & 3+8\\15+8 & 9+16\end{bmatrix}=\begin{bmatrix}9 & 11\\23 & 25\end{bmatrix}\ldots\ldots\text{(ii)}$$

$$AC=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}6 & 2\\5 & 1\end{bmatrix}=\begin{bmatrix}1\times 6+2\times 5 & 1\times 2+2\times 1\\36+4\times 5 & 3\times 2+4\times 1\end{bmatrix}$$

$$=\begin{bmatrix}6+10 & 2+2\\18+20 & 6+4\end{bmatrix}=\begin{bmatrix}16 & 4\\38 & 10\end{bmatrix}\ldots\ldots\text{(iii)}$$

Form (ii) and (iii),

$$AB+AC=\begin{bmatrix}9 & 11\\23 & 25\end{bmatrix}+\begin{bmatrix}16 & 4\\38 & 10\end{bmatrix}=\begin{bmatrix}9+16 & 11+4\\23+38 & 25+10\end{bmatrix}$$

$$=\begin{bmatrix}25 & 15\\61 & 35\end{bmatrix}\ldots\ldots\text{(iv)}$$

$\therefore$ Form (i) and (iv)

$$A(B+C)=AB+AC$$

Practice:

Verify that

$$(A+B)C=AC+BC$$

2.4.5 Multiplicative identity of a matrix

Let we have a matrix I and a matrix A. If these two matrices are so related to each other such that $IA=AI=A$ i.e., the multiplication of I with A or A with I doesn't change the value of A then I is called the identity matrix of A. For example:

$Let\ A=\begin{bmatrix}1 & 2\\4 & 5\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ then we see that

$$IA=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\4 & 5\end{bmatrix}$$

$$=\begin{bmatrix}1\times 1+0\times 4 & 1\times 2+0\times 5\\0\times 1+1\times 4 & 0\times 2+1\times 5\end{bmatrix}$$

$$=\begin{bmatrix}1+0 & 2+0\\0+4 & 0+5\end{bmatrix}$$

$$IA=\begin{bmatrix}1 & 2\\4 & 5\end{bmatrix}=A$$

Therefore, $IA=A$

AI is also defined because number of columns in $A=2$ and number of rows in I is also 2.

So, $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is a

multiplicative identity of $A=\begin{bmatrix}1 & 2\\4 & 5\end{bmatrix}$

Example 2.12:

If $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $A=\begin{bmatrix}9 & -3\\-1 & 5\end{bmatrix}$, then find IA and AI.

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Solution:

$$IA=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}9 & -3\\-4 & 5\end{bmatrix}$$

$$=\begin{bmatrix}1\times 9+0\times(-4) & 1\times(-3)+0\times 5\\0\times 9+1\times(-4) & 0\times(-3)+1\times 5\end{bmatrix}$$

$$IA=\begin{bmatrix}9 & -3\\-4 & 5\end{bmatrix}=A$$

$$AI=\begin{bmatrix}9 & -3\\-4 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}9\times 1+(-3)+0 & 9\times 0+(-3)\times 1\\(-4)\times 1+5\times 0 & -4\times 0+5\times 1\end{bmatrix}$$

$$IA=\begin{bmatrix}9 & -3\\-4 & 5\end{bmatrix}=A$$

Hence, $IA=AI=A$

So $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is the multiplicative identity of $A=\begin{bmatrix}9 & -3\\-1 & 5\end{bmatrix}$.

In fact $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is the multiplicative identity of the system of all **2 by 2 square matrices.**

Similarly $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ is the multiplicative identity of the system of all 3 by 3 square matrices.

2.4.6 Transpose of a matrix

A matrix which is obtained by interchanging all the rows and columns of a given matrix is called its transpose. The transpose of matrix A is written A^t.

Example 2.13:

If $A=\begin{bmatrix}3 & 4 & 5\\2 & 4 & 6\end{bmatrix}$, then $A^t=\begin{bmatrix}3 & 2\\4 & 4\\5 & 6\end{bmatrix}$

2.4.7 Verification of the result $(AB)^t=B^tA^t$

For verifying the result $(AB)^t=B^tA^t$ it is necessary that must exists i.e. A & B are conformable for multiplication. This is possible only, when number of columns in A must be equal to the number of rows in B. Now we choose matrices of suitable order/orders so that their multiplication becomes possible and then try to verity the above law, $(AB)^t=B^tA^t$ that is, the transpose of the product of matrices is equal to the product of their transposes but in the reverse order.

Example 2.14:

Let $A=\begin{bmatrix}3 & -1\\1 & 4\end{bmatrix}, B=\begin{bmatrix}2 & -5\\6 & -7\end{bmatrix}$, show that $(AB)^t=B^tA^t$.

Solution:

$$AB=\begin{bmatrix}3 & -2\\1 & 4\end{bmatrix}\begin{bmatrix}2 & -5\\6 & -7\end{bmatrix}$$

$$=\begin{bmatrix}3\times 2+(-2)\times 6 & 3\times(-5)+(-2)(-7)\\1\times 2+4\times 6 & 1\times(-5)+4\times(-7)\end{bmatrix}$$

$$=\begin{bmatrix}6-12 & -15+14\\2+24 & -5-28\end{bmatrix}=\begin{bmatrix}-6 & -1\\26 & -33\end{bmatrix}\ \ldots\ldots\text{(i)}$$

$$B^tA^t=\begin{bmatrix}2 & 6\\-5 & -7\end{bmatrix}\begin{bmatrix}3 & 1\\-2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}2\times 3+6\times(-2) & 2\times 1+6\times 4\\-5\times 3+(-7)(-2) & (-5)\times 1+(-7)\times 4\end{bmatrix}$$

$$=\begin{bmatrix}6-12 & 2+24\\-15+14 & -5-28\end{bmatrix}=\begin{bmatrix}-6 & 26\\-1 & -33\end{bmatrix}\ \ldots\ldots\text{(ii)}$$

Form (i) and (ii) we have $(AB)^t=B^tA^t$

Skill 2.4

✧ **Matrix Operations:** Familiarity with the rules and principles for performing multiplication .

Exercise 2.4

1. Show that which of the following matrices are conformable for multiplication.

   i) $A=\begin{bmatrix}a\\b\end{bmatrix}$ **ii)** $B=\begin{bmatrix}p & q\end{bmatrix}$

   iii) $C=\begin{bmatrix}1 & -1\\-2 & 1\end{bmatrix}$ **iv)** $D=\begin{bmatrix}p & r & s\end{bmatrix}$

2. If $A=\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}, B=\begin{bmatrix}3\\-2\end{bmatrix}$

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i) Is it possible to find AB? **ii)** Is it possible to find BA? **iii)** Find the possible product/products.

3. Given that

$$A=\begin{bmatrix}4&1\\3&1\end{bmatrix}, B=\begin{bmatrix}1&-1\\-3&4\end{bmatrix}, C=\begin{bmatrix}3&4\\1&2\end{bmatrix}, D=\begin{bmatrix}1&-2\\-\frac{1}{2}&\frac{2}{3}\end{bmatrix}$$

Find **i)** AB and **ii)** CD.

4. Let $A=\begin{bmatrix}2&1\\3&0\\-1&4\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\2&1\end{bmatrix}$. **i)** Find AB.

ii) Does BA exist?

5. If $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}, B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, then show that $AB \neq BA$.

6. If $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, then find $A\times A$.

7. If $A=\begin{bmatrix}-2&3\\2&-1\end{bmatrix}, B=\begin{bmatrix}1&-1\\2&4\end{bmatrix}$. Is $AB=BA$.

8. If $A=\begin{bmatrix}-1\\1\end{bmatrix}, B=\begin{bmatrix}1&-2\end{bmatrix}, C=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$, then find

i) $(AB)C$ and $A(BC)$.

ii) Determine whether $(AB)C=A(BC)$.

iii) Interpret which law of multiplication this result shows?

9. Verify that $A(B+C)=AB+AC$ for the following matrices.

i) $A=\begin{bmatrix}1&2\\3&-1\end{bmatrix}, B=\begin{bmatrix}1&0\\0&2\end{bmatrix}, C=\begin{bmatrix}3&-1\\0&2\end{bmatrix}$

ii) $A=\begin{bmatrix}3&-1\\0&2\end{bmatrix}, B=\begin{bmatrix}1\\2\end{bmatrix}, C=\begin{bmatrix}-1\\1\end{bmatrix}$

10. Let $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}, A=\begin{bmatrix}5&-3\\4&6\end{bmatrix}, B=\begin{bmatrix}-7&3\\2&8\end{bmatrix}$ Find

i) AI **ii)** BI

11. Prove that

i) $A=\begin{bmatrix}3&2&1\end{bmatrix}, B=\begin{bmatrix}-3&4&2\end{bmatrix}$,

$(A+B)^t=A^t+B^t$ and $(A-B)^t=A^t-B^t$

ii) $C=\begin{bmatrix}7&-3\\2&-1\end{bmatrix}, D=\begin{bmatrix}1&1\\2&2\end{bmatrix}$,

$(C+D)^t=C^t+D^t$ and $(C-D)^t=C^t-D^t$

12. i) If $A=\begin{bmatrix}2&5\\-3&4\end{bmatrix}, B=\begin{bmatrix}-1&1\\2&3\end{bmatrix}$, show that $(AB)^t=B^tA^t$

ii) If $C=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, show that $\left(C^t\right)^t=C$

iii) If $A=\begin{bmatrix}1&0&-1\\2&0&6\end{bmatrix}, B=\begin{bmatrix}1&7\\-8&4\\0&1\end{bmatrix}$, show that $(AB)^t=B^tA^t$

Student Learning Outcomes

- Evaluate the determinant and inverse of a matrix of order 2 × 2.

2.5 MULTIPLICATIVE INVERSE OF A MATRIX

Similar to finding the reciprocal of a number, matrices also have a concept of a multiplicative inverse. However, unlike simple division, matrices follow a specific procedure to determine their inverse due to the absence of conventional division in matrix operations. Unveiling this unique approach allows us to unlock new possibilities in matrix algebra

2.5.1 Determinant of a square matrix

With every square matrix A, a unique real number is associated which is called the determinant of A denoted by $|A|$ or det A and is given by a definite rule.

Thus if $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $|A|=\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$

Thus the determinant of a 2-by-2 matrix is obtained by multiplying entries on the main diagonal and

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subtracting from it the product of entries lying on the secondary diagonal.

Example 2.15:

Find the determinant of the matrix

$$A=\begin{bmatrix}7 & 5\\ 17 & -12\end{bmatrix}$$

Solution:

Given $A=\begin{bmatrix}7 & 5\\ 7 & -12\end{bmatrix}$

$$\therefore\ |A|=\begin{vmatrix}7 & 5\\ 7 & -12\end{vmatrix}=7\times(-12)-5\times 7=-84-35=-119$$

2.5.2 Singular and non-singular matrices

A square matrix A is called singular if $|A|=0$ and non-singular if $|A|\neq 0$.

Example 2.16:

Check whether $A=\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$ is a singular matrix or not.

Solution:

If $A=\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$, then

$$|A|=\begin{vmatrix}4 & -2\\ -2 & 1\end{vmatrix}=(4)(1)-(-2)(-2)=4-4=0$$

Hence A is a singular matrix.

Example 2.17:

If $P=\begin{bmatrix}-4 & 2\\ 3 & -7\end{bmatrix}$, check whether P is a singular or non-singular matrix.

Solution:

$$P=\begin{bmatrix}-4 & 2\\ 3 & -7\end{bmatrix},\ then\ |P|=\begin{vmatrix}-4 & 2\\ 3 & -7\end{vmatrix}$$

$$=(-4)\times(-7)-(3)\times(2)$$

$$=28-6=22\neq 0$$

Since $|P|\neq 0$, therefore P is non-singular matrix.

2.5.3 Adjoint of a matrix

The adjoint of a square matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is denoted by $adjA$ and defined as $adjA=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$.

That is, change the places of a and d with each other and change the signs of b and c.

Example 2.18:

Find adjoint of the following matrices.

i) $A=\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}$ **ii)** $B=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$

Solution:

i) $A=\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}$, then

$$adjA=\begin{bmatrix}4 & 2\\ -1 & 3\end{bmatrix}$$

ii) $B=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$, then

$$adjB=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$$

2.5.4 Multiplicative inverse of a Matrix

Let A be a non-singular square matrix. If there exists another non-singular matrix B such that $AB=BA=I$ is an identity matrix, then B is said to be the multiplicative inverse of A. we denote the inverse of A b.y A^{-1}. Hence $B=A^{-1}$.

Example 2.19:

Show that $\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}$ is the multiplicative inverse of $\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}$.

Solution: $Let\ A=\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}, B=\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}$

$$AB=\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}3\times 3+2(-4) & 3(-2)+2\times 3\\ 4\times 3+3(-4) & 4(-2)+3\times 3\end{bmatrix}$$

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$$=\begin{bmatrix} 9-8 & -6+6 \\ 12-12 & -8+9 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$BA=\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}=\begin{bmatrix} (3\times3)+(-2)4 & 3\times2+(-2)3 \\ (-4\times3)+(3\times4) & (-4)2+3\times3 \end{bmatrix}$$

$$=\begin{bmatrix} 9-8 & -6+6 \\ -12+12 & -8+9 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

Since $AB=I=BA$, therefore A is the inverse of B.

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $Adj(A)=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $|A|=ad-bc$.

To find A^{-1}, we use the formula $A^{-1}=\frac{1}{|A|}\,Adj\,A$.

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.20:

Find the inverse of $A=\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$, using the adjoint method.

Solution: $A=\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$

$$|A|=-2\times4-3(-1)=-8+3=-5$$

Since $|A|\neq 0$, therefore A is non-singular, so we can find A^{-1}.

$$A^{-1}=\frac{1}{|A|}AdjA$$

Now $AdjA=\begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}$, putting it in equation (1).

$$\therefore\ A^{-1}=\frac{1}{-5}\begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}=\begin{bmatrix} \frac{-4}{5} & \frac{-1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

2.5.6 Verification of the result $AA^{-1}=I=A^{-1}A$

$$AA^{-1}=\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{-4}{5} & \frac{-1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$=\begin{bmatrix} -2\times\left(\frac{-4}{5}\right)+(-1)\times\left(\frac{3}{5}\right) & -2\times\left(\frac{-1}{5}\right)+(-1)\times\left(\frac{2}{5}\right) \\ 3\times\left(\frac{-4}{5}\right)+4\times\left(\frac{3}{5}\right) & 3\times\left(\frac{-1}{5}\right)+4\times\frac{2}{5} \end{bmatrix}$$

$$\begin{bmatrix} \frac{8}{5}-\frac{3}{5} & \frac{2}{5}-\frac{2}{5} \\ \frac{-12}{5}+\frac{12}{5} & \frac{-3}{5}+\frac{8}{5} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I \quad \ldots\ldots\ (i)$$

$$A^{-1}A=\begin{bmatrix} \frac{-4}{5} & \frac{-1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} \left(\frac{-4}{5}\right)\times(-2)+\left(\frac{-1}{5}\right)\times(3) & \left(\frac{-4}{5}\right)\times(-1)+\left(\frac{-1}{5}\right)\times(4) \\ \left(\frac{3}{5}\right)\times(-2)+\left(\frac{2}{5}\right)\times(3) & \left(\frac{3}{5}\right)\times(-1)+\left(\frac{2}{5}\right)\times(4) \end{bmatrix}$$

$$A^{-1}A=\begin{bmatrix} \frac{8}{5}-\frac{3}{5} & \frac{4}{5}-\frac{4}{5} \\ \frac{-6}{5}+\frac{6}{5} & \frac{-3}{5}+\frac{8}{5} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \ldots\ldots\ (ii)$$

Thus, from (i) and (ii)

$$AA^{-1}=I=A^{-1}A.$$

2.5.7 Verification of the result $(AB)^{-1}=B^{-1}A^{-1}$

The above equation is not true for all types of matrices, but it is true only if A and B are non-singular square matrices of the same order. In such a case, the above equation states that the inverse of the product matrix is equal to the product of inverses but taken in the reverse order.

Now we verify the above result by taking suitable matrices.

Example 2.21:

Let $A=\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

Solution:

$$AB=\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}=\begin{bmatrix} -2\times2+1\times3 & -2\times1+1\times2 \\ 1\times2+1\times3 & 1\times1+1\times2 \end{bmatrix}$$

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$$=\begin{bmatrix}-4+3 & -2+2\\ 2+3 & 1+2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 5 & 3\end{bmatrix}$$

$$\therefore\ det(AB)=\begin{vmatrix}-1 & 0\\ 5 & 3\end{vmatrix}=(-1)\times(3)-(0)\times(5)=-3\neq 0$$

$\therefore\ (AB)^{-1}$ exists and $(AB)^{-1}=\dfrac{1}{det(AB)}Adj(AB)$

Since, $AB=\begin{bmatrix}-1 & 0\\ 5 & 3\end{bmatrix}\Rightarrow Adj(AB)=\begin{bmatrix}3 & 0\\ -5 & -1\end{bmatrix}$

$$(AB)^{-1}=-\frac{1}{3}\begin{bmatrix}3 & 0\\ -5 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{3}\times 3 & -\frac{1}{3}\times 0\\ -\frac{1}{3}\times -5 & -\frac{1}{3}\times(-1)\end{bmatrix}=\begin{bmatrix}-1 & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}\ \ldots\ldots\ (i)$$

Next,

$$A=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}\Rightarrow det(A)=\begin{vmatrix}-2 & 1\\ 1 & 1\end{vmatrix}=(-2)(1)-(1)(1)$$

$$detA=-2-1=-3\neq 0$$

$$\therefore\ Adj(A)=\begin{bmatrix}1 & -1\\ -1. & -2\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{detA}Adj(A)=-\frac{1}{3}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

$$i.e., A^{-1}=\begin{bmatrix}-\frac{1}{3}\times 1 & -\frac{1}{3}\times(-1)\\ -\frac{1}{3}\times(-1) & -\frac{1}{3}\times(-2)\end{bmatrix}=\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

Again, $B=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$ and $|B|=\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}=2(2)-1(3)$,

so B^{-1} exists.

$B^{-1}=\dfrac{1}{|B|}Adj(B)$. But $Adj(B)=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$

$$\therefore\ B^{-1}=\frac{1}{1}\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}$$

$$\therefore\ B^{-1}A^{-1}=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

$$=\begin{bmatrix}2\times-\frac{1}{3}+(-1)\times\frac{1}{3} & 2\times\frac{1}{3}+(-1)\times\frac{2}{3}\\ -3\times\left(-\frac{1}{3}\right)+2\times\frac{1}{3} & -3\times\frac{1}{3}+2\times\frac{2}{3}\end{bmatrix}$$

$$B^{-1}A^{-1}=\begin{bmatrix}-\frac{2}{3}-\frac{1}{3} & \frac{2}{3}-\frac{2}{3}\\ 1+\frac{2}{3} & -1\times\frac{4}{3}\end{bmatrix}=\begin{bmatrix}\frac{-2-1}{3} & 0\\ \frac{3+2}{3} & \frac{-3+4}{3}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-3}{3} & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}$$

$$\text{i.e., } B^{-1}A^{-1}=\begin{bmatrix}-1 & 0\\ \frac{5}{3} & \frac{1}{3}\end{bmatrix}\ \ldots\ldots\ (ii)$$

Comparing (i) and (ii) we get $(AB)^{-1}=B^{-1}A^{-1}$.

Skill 2.5

Calculating Determinants and Inverses: Proficiency in determining the determinant and inverse of 2×2 matrices and interpreting their significance.

Exercise 2.5

1. Find the determinant of following matrices and evaluate them.

i) $A=\begin{bmatrix}5 & 6\\ -4 & 1\end{bmatrix}$ **ii)** $B=\begin{bmatrix}4 & -2\\ 5 & 13\end{bmatrix}$

iii) $C=\begin{bmatrix}11 & 7\\ -6 & 5\end{bmatrix}$ **iv)** $D=\begin{bmatrix}5 & 6\\ -8 & -9\end{bmatrix}$

v) $E=\begin{bmatrix}2p & -3q\\ r & -s\end{bmatrix}$ **vi)** $F=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

vii) $G=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}$ **viii)** $H=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$

2. Find which of the following matrices are singular and which are nonsingular.

i) $A=\begin{bmatrix}5 & 3\\ 2 & 1\end{bmatrix}$ **ii)** $B=\begin{bmatrix}3 & -6\\ -2 & 4\end{bmatrix}$

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iii) $C=\begin{bmatrix}3a & -2b\\ 2a & b\end{bmatrix}$ **iv)** $D=\begin{bmatrix}-3 & 6\\ 2 & -4\end{bmatrix}$

3. **Find the adjoint of the following matrices.**

i) $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ **ii)** $B=\begin{bmatrix}-3 & -1\\ 2 & 3\end{bmatrix}$

iii) $C=\begin{bmatrix}2 & -4\\ 3 & 1\end{bmatrix}$ **iv)** $D=\begin{bmatrix}-3 & 6\\ 2 & -4\end{bmatrix}$

4. **Find the multiplicative inverses of the following matrices if they exist.**

i) $A=\begin{bmatrix}4 & 1\\ 3 & 1\end{bmatrix}$ **ii)** $B=\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$

iii) $C=\begin{bmatrix}4 & -3\\ -1 & 2\end{bmatrix}$ **iv)** $D=\begin{bmatrix}0 & -3\\ 2 & 4\end{bmatrix}$

v) $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

5. If $A=\begin{bmatrix}2 & 0\\ -3 & 1\end{bmatrix}, B=\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}$ find

i) AB **ii)** BA **iii)** A^{-1} and B^{-1}

then show that $(AB)^{-1}=B^{-1}A^{-1}$ and $(BA)^{-1}=A^{-1}B^{-1}$

6. If $A=\begin{bmatrix}0 & -1\\ 2 & 1\end{bmatrix}, B=\begin{bmatrix}2 & 3\\ 1 & 0\end{bmatrix}$, then show that

$(AB)^{-1}=B^{-1}A^{-1}$ and $(BA)^{-1}=A^{-1}B^{-1}$

Student Learning Outcomes

- Solve the simultaneous linear equations in two variables using matrix inversion method and Cramer's rule

2.6 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

2.6.1 Matrix Inversion Method

System of two linear equations in variable x and y in general form is shown below.

$$ax+by=m \;\cdots\cdots\; \text{(i)}$$

$$cx+dy=n \;\cdots\cdots\; \text{(ii)}$$

This can be written as, $AX=B \cdots\cdots$ (iii)

where $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, and $B=\begin{bmatrix}m\\ n\end{bmatrix}$.

So, the system of equations in matrix form is,

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}m\\ n\end{bmatrix}$$

If $|A|\neq 0$, we can find A^{-1}. Multiplying both sides of equation (iii) by A^{-1}, we get

$A^{-1}(AX)=A^{-1}B \Rightarrow (A^{-1}A)X=A^{-1}B$

$\Rightarrow IX=A^{-1}B \Rightarrow X=A^{-1}B \cdots\cdots$ (iv)

From equation (iv),

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\begin{bmatrix}m\\ n\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} \; or \; \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}dm-bn\\ -cm+an\end{bmatrix}$$

$$\Rightarrow x=\frac{dm-bn}{ad-bc},\; y=\frac{-cm+an}{ad-bc}$$

Hence, the solution of given system of equations is

$$x=\frac{dm-bn}{ad-bc},\; y=\frac{-cm+an}{ad-bc}$$

This method is known as matrix inversion method. It may be remembered that if $|A|=0, A^{-1}$ does not exist. Hence, in such a case solution set of the equations cannot be found.

Example 2.22:

Solve the following system of equations with the help of matrices.

Solution:

$$x-3y=0,\; 2x+y=7$$

These equations can be written in the form of matrices as

$$\begin{bmatrix}1 & -3\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 7\end{bmatrix}$$

Let $A=\begin{bmatrix}1 & -3\\ 2 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, $B=\begin{bmatrix}0\\ 7\end{bmatrix}$

$$|A|=1\times 1-2(-3)-1+6=7\neq 0.$$

Hence, A^{-1} exists. By using formula, we get

$$A^{-1}=\frac{1}{7}\begin{bmatrix}1 & 3\\ -2 & 1\end{bmatrix}$$

Now, $AX=B \Rightarrow X=A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{7}\begin{bmatrix}1 & 3\\ -2 & 1\end{bmatrix}\begin{bmatrix}0\\ 7\end{bmatrix}$

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$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1\times 0+3\times 7 \\ -2\times 0+1\times 7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 21 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\therefore \ x=3,\ y=1$$

Example 2.23:

Is the following system of equations solvable?

$$3x-6y=9,\ 2x-4y=-3$$

Solution:

Here $A=\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}$,

then $|A|=\begin{vmatrix} 3 & -6 \\ 2 & -4 \end{vmatrix}=3(-4)-2(-6)=-12+12=0$

Hence the given equations are non-solvable.

2.6.2 Cramer's Rule

We can easily solve simultaneous equations by applying Cramer's rule. The method is explained as:

Let the system of equations be

$$ax+by=m$$
$$cx+dy=n$$

In terms of matrices, we write these equations as;

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} m \\ n \end{bmatrix},\ \text{or } A\begin{bmatrix} x \\ y \end{bmatrix}=B,$$

where $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} m \\ n \end{bmatrix}$.

$$|A|=ad-bc$$

We replace the coefficients of x in A (that is a,c) by m,n of B and denote as A_x.

$$\therefore \quad A_x=\begin{bmatrix} m & b \\ n & d \end{bmatrix}, |A_x|=md-nb,$$

$$\text{then } \frac{|A_x|}{|A|}=\frac{md-nb}{ad-bc}$$

Likewise, for y, we replace the coefficients y in A (that is b,d) by m,n of B and denote it by A_y.

$$A_y=\begin{bmatrix} a & m \\ c & n \end{bmatrix} \ where |A_y|=an-mn,$$

$$\text{then } \ y=\frac{|A_y|}{|A|}=\frac{an-cm}{ad-bc}$$

The method explained above is known as Cramer's Rule.

Example 2.24:

Solve the following system of equations by using Cramer's rule.

$$x-2y=1,\ 3x+y=10$$

Solution:

In terms of matrices we can write the above system as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 10 \end{bmatrix} \Rightarrow AX=B$$

where $A=\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, B=\begin{bmatrix} 1 \\ 10 \end{bmatrix}$

$$|A|=1\times 1-3(-2)=1+6=7\neq 0$$

Replacing the coefficients of x in A (that is 1,3) by 1,10 of B to find A_x.

$$A_x=\begin{bmatrix} 1 & -2 \\ 10 & 1 \end{bmatrix}, \text{giving } |A_x|=1\times 1-10(-2)$$
$$=1+20=21$$

$$\therefore \ x=\frac{|A_x|}{A}=\frac{21}{7}=3$$

For A_y, replace the coefficients of y in A (that is $-2,1$) by 1,10 of B.

Then $A_y=\begin{bmatrix} 1 & 1 \\ 3 & 10 \end{bmatrix}$, giving

$$|A_y|=1\times 10-3(1)=10-3=7$$

$$\therefore \ y=\frac{|A_y|}{A}=\frac{7}{7}=1$$

Hence the solution set $=\{(3,1)\}$

Student Learning Outcomes

- Explain, with examples, how mathematics plays a key role in the development of new scientific theories and technologies. [e.g., Mathematical models and simulations are used to design and optimize new materials and drugs, and to understand the behavior of complex systems such as the human brain.]

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✧ Apply concepts of matrices to real world problems (such as engineering, economics, computer graphics, and physics)

2.6.3 Real life problems leading to simultaneous equations

Many real life problems require for their solution the finding of two unknown quantities. The general method of solution is similar to that used when there is one unknown, but with the important difference that when there are two unknowns to be found, two equations must be formed form the data, which can be solved by either matrix inversion method or Cramer's rule as shown in the following examples.

Example 2.25:

My friend asked me this question. There are two numbers such that the sum of the first and three times the second is 53, while the difference between 4 times the first and twice the second is 2. Can you help me out in finding the numbers?

Solution:

Let $x =$ one number and $y =$ the second.

Then from the first set of facts

$$x + 3y = 53$$

From the second set of facts

$$4x - 2y = 2$$

These equations can be written in the form of matrices as:

$$\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 35 \\ 2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 53 \\ 2 \end{bmatrix}$

Now $|A| = 1\times(-2) - 3\times 4 = -2 - 12 = -14 \neq 0$. Hence A^{-1} exists. By using formula we get

$$A^{-1} = \frac{1}{-14}\begin{bmatrix} -2 & -3 \\ -4 & 1 \end{bmatrix} = -\frac{1}{14}\begin{bmatrix} -2 & -3 \\ -4 & 1 \end{bmatrix}$$

Now $AX = B \Rightarrow A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{14}\begin{bmatrix} -2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 53 \\ 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{14}\begin{bmatrix} -2\times 53 + (-3)\times 2 \\ -4\times 53 + 1\times 2 \end{bmatrix}$$

$$= -\frac{1}{14}\begin{bmatrix} -106 - 6 \\ -212 + 2 \end{bmatrix} = -\frac{1}{14}\begin{bmatrix} -112 \\ -210 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

$$\therefore\ x = 8,\ y = 15$$

Therefore, the numbers are 8 and 15 .

Example 2.26:

The cost of 1 rubber and 7 sharpeners are 8 rupees, while that of 3 rubbers and 1 sharpener are 3 rupees. What are the prices of a rubber and a sharpener respectively?

Solution:

Let $x =$ rubber price.

$y =$ sharpener price.

Then from the first set of facts

$$x + 7y = 15$$

From the second,

$$3x + y = 5$$

These equations can be written in the form of matrices as:

$$\begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

Now $D = |A| = 1\times 1 - 3\times(7) = 1 - 21 = -20 \neq 0$.

Here $A_x = \begin{bmatrix} 15 & 7 \\ 5 & 1 \end{bmatrix}$, giving

$$D_1 = |A_x| = 15\times 1 - 5(7) = 15 - 35 = -20$$

$$\therefore\ x = \frac{D_1}{D} = \frac{-20}{-20} = 1$$

$A_y = \begin{bmatrix} 1 & 15 \\ 3 & 5 \end{bmatrix}$, *giving* $D_2 = |A_y| = 1\times 5 - 3(15) = 5 - 45 = -40$

$$\therefore\ x = 1, y = 2$$

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Thus price of one rubber $= \therefore\ y = \frac{D_2}{D} = \frac{-40}{-20} = 21$ rupee and price of one sharpener $= 2$ rupees.

Skill 2.6

Capability to use matrix inversion and Cramer's rule to find solutions to systems of linear equations Solving Linear Systems

Exercise 2.6

1. Solve the following system of linear equations using inversion method.
   - **i)** $2x+3y=-1 \quad ; \quad x-y=2$
   - **ii)** $x+2y=-13 \quad ; 3x+6y=11$
   - **iii)** $x+2y=1 \qquad ; 2x+3y=\frac{5}{2}$
   - **iv)** $x-2y-1=0;\ 2x+y+3=0$
2. Solve the following system of linear equations using Cramer's Rule.
   - **i)** $x-2y=5 \qquad ; 2x-y=6$
   - **ii)** $4x+3y=-2 \quad ; \quad x-2y=5$
   - **iii)** $5x+7y=3 \quad ; \quad 3x+y=5$
3. Amjad thought of two numbers whose sum is 12 and whose difference is 4 . Find the numbers.
4. The length of a rectangular playground is twice its width. The perimeter is 30. Find its dimensions.
5. 3 bags and 4 pens together cost 257 rupees whereas 4 bags and 3 pens together cost 324 rupees. Find the cost of a bag and 10 pens.
6. If twice the son's age in years is added to the father's age, the sum is 70 . But if the father's age is added to the son's age, the sum is 95 . Find the ages of father and son.

2.7 APPLICATIONS IN DEVELOPMENT OF NEW SCIENTIFIC THEORIES AND TECHNOLOGIES IN REAL WORLD

Matrices play a vital role in modeling dynamic systems and analyzing large data sets. They simplify complex differential equations into a manageable form for simulations in engineering, helping predict system behavior. In research and pharmaceuticals, matrices are used to process and interpret experimental data, aiding in material development and drug efficacy studies.

Matrices are key tools in various fields, essential for solving real world problems. In engineering, they help design structures and analyze safety. In economics, they model market dynamics and equilibrium. In computer graphics, matrices facilitate 2D and 3D transformations, crucial for rendering images and scenes. They also play a significant role in image processing tasks like blurring and sharpening.

Example 2.27:

Consider a mass-spring system with a single degree of freedom described by the following matrix equation:

$$X_t = AX_0$$

Here, X_t represents the state of system of at time t, X_0 represents the initial state of the system and the A is a matrix that represents the system's dynamics. Both X_t and X_0 describe the state of the system by containing the values for position and velocity of the system at a particular time. The matrix A describes the system's dynamics by containing the physical constants for the system such as the stiffness co-efficient.

Now calculate the initial state of the system (X_0) if:

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}$$

$$X_t = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \ at\ t=2$$

$$k = \ stiffness\ coefficient = 2$$

$$c = \ damping\ coefficient = 0.5$$

This problem can be solved using the matrix inversion method:

$$X_t = AX_0$$

$$A^{-1}X_t = X_0$$

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By using the values of the constants, we have the A matrix as:

$$A=\begin{bmatrix}0 & 1\\ -2 & -0.5\end{bmatrix}$$

To find A^{-1} we need to calculate both the determinant and Adjoint of A:

$$|A|=\begin{vmatrix}0 & 1\\ -2 & -0.5\end{vmatrix}=(0)(-0.5)-(-2)(1)=2$$

$$AdjA=\begin{bmatrix}-0.5 & -1\\ 2 & 0\end{bmatrix}$$

Uses

In 1939, Britain enlisted chess players, mathematicians, and logicians to decipher Nazi codes during World War II. Over 10,000 people collaborated, successfully breaking the code within a year. This section explores how matrices and their inverses played a crucial role in concealing and revealing hidden messages

Now the inverse can be calculated as:

$$A^{-1}=\frac{1}{|A|}AdjA$$

$$A^{-1}=\frac{1}{2}\begin{bmatrix}-0.5 & -1\\ 2 & 0\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-1 & -2\\ 4 & 0\end{bmatrix}$$

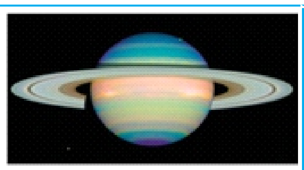
Finally, we can now calculate X_0 by matrix multiplication: $A^{-1}X_t=X_0$

$$\begin{bmatrix}-1 & -2\\ 4 & 0\end{bmatrix}\begin{bmatrix}-2\\ 0\end{bmatrix}=X_0$$

$$\begin{bmatrix}(-1)(-2)+(-2)(0)\\ (4)(-2)+(0)(0)\end{bmatrix}=X_0$$

$$\begin{bmatrix}2\\ -8\end{bmatrix}=X_0$$

Uses



Photographs sent back from space use matrices with thousands of pixels. Each pixel is assigned a number from 0 to 63 representing its color—0 for pure white and 63 for pure black. In the image of Saturn shown here, matrix operations provide false colors that emphasize the banding of the planet's upper atmosphere

Example 2.28:

A chemical reaction involving two reactants and one product is described by the matrix equation:

$$C_{eq}=KC_i$$

Where, C_{eq} is a column matrix representing the concentrations of reactants and products at equilibrium, K is a matrix describing the reaction rate constants and C_i is a column matrix representing the initial concentrations of reactants and products.

Now calculate the value of C_{eq} if:

$$C_i=\begin{bmatrix}1\\ 3\\ 2\end{bmatrix}$$

$$K=\begin{bmatrix}-0.1 & 0 & 0.2\\ 0 & 0.1 & 0\\ 0 & 0.3 & -0.2\end{bmatrix}$$

Solution:

This problem can be simply solved by matrix multiplication:

$$C_{eq}=KC_i$$

$$C_{eq}=\begin{bmatrix}-0.1 & 0 & 0.2\\ 0 & 0.1 & 0\\ 0 & 0.3 & -0.2\end{bmatrix}\begin{bmatrix}1\\ 3\\ 2\end{bmatrix}$$

$$C_{eq}=\begin{bmatrix}(-0.1)(1)+(0)(3)+(0.2)(2)\\ (0)(1)+(0.1)(3)+(0)(2)\\ (0)(1)+(0.3)(3)+(-0.2)(2)\end{bmatrix}$$

$$C_{eq}=\begin{bmatrix}0.3\\ 0.3\\ 0.5\end{bmatrix}$$

Example 2.29:

A bridge is subjected to a force of 1000 Newton. The stress in the bridge can be calculated using the following matrix equation:

$S = A\times B\times$ force

where $A=\begin{bmatrix}1000 & 500\\ 500 & 2000\end{bmatrix}$ and, $B=\begin{bmatrix}2000\\ 3000\end{bmatrix}$

(A is a matrix representing the properties of the bridge, B represent he force applied to the bridge and force is a scalar value representing the magnitude of the force and S is matrix representing the stress in a bridge)

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Solution: $S = A \times B \times$ force

$$S = \begin{bmatrix} 1000 & 500 \\ 500 & 2000 \end{bmatrix} \begin{bmatrix} 2000 \\ 3000 \end{bmatrix} \times 1000$$

$$S = \begin{bmatrix} 1000 & 500 \\ 500 & 2000 \end{bmatrix} \begin{bmatrix} 2000 \times 1000 \\ 3000 \times 1000 \end{bmatrix}$$

$$S = \begin{bmatrix} 1000 & 500 \\ 500 & 2000 \end{bmatrix} \begin{bmatrix} 2000000 \\ 3000000 \end{bmatrix}$$

$$S = \begin{bmatrix} 1000(2000000) + 500(3000000) \\ 500(2000000) + 2000(3000000) \end{bmatrix}$$

$$S = \begin{bmatrix} 3500000000 \\ 9500000000 \end{bmatrix}$$

So, the stress in a bridge is given by matrix **S**

Example 2.30:

Consider a network of parallel resistors described by the system of equations: $3x - 2y = 6; 2x + 4y = 8$

Represent these relationships in a matrix and solve for x and y.

Solution:

This problem can be solved using the matrix inversion method. The matrix form of this system is $AX = B$,

where $A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

The above equation can be written as

$$X = A^{-1} B$$

$$X = \frac{AdjA}{|A|} \times B$$

Firstly, we need to calculate both the determinant of

$$|A| = \begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix}$$

$$|A| = (3)(4) - (-2)(2)$$

$$|A| = 12 + 4$$

$$|A| = 16$$

The adjoint of A will be:

$$AdjA = \begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

Now values of adjoint of A and determinant of A in above equation

$$X = \frac{AdjA}{|A|} \times B \quad X = \frac{\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}}{16} \times \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$X = \frac{\begin{bmatrix} (4)(6) + (2)(8) \\ (-2)(6) + (3)(8) \end{bmatrix}}{16} \quad X = \frac{\begin{bmatrix} 40 \\ 12 \end{bmatrix}}{16} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40/16 \\ 12/16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/4 \end{bmatrix}$$

So, the solution to the system of equations is $x = 5/2$ and $y = 3/4$

Example 2.31:

Consider a 2D point represented by the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. The transformation matrix for scaling by a factor of 2 in the x -direction and 3 in the y-direction is:

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

a) Apply the transformation matrix T to the point

$$A = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

b) What is the result of applying the transformation T twice?

Solution:

a) To apply the transformation matrix to the point, simply multiply the matrix T by the column vector representing the point:

$$T \times A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(4) + 0(2) \\ 0(4) + 3(2) \end{bmatrix}$$

b) To apply the transformation matrix T twice to the point $A = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, simply perform the multiplication as

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$$T \times T \times A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2(4)+0(2) \\ 0(4)+3(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 18 \end{bmatrix}$$

Skill 2.7

- **Explaining Mathematical Applications:**
  Ability to articulate and exemplify the role of mathematics in the development of new scientific theories and technologies.
- **Practical Application of Matrix Concepts:**
  Aptitude for applying matrix theory to solve real-world problems in various professional and academic fields

Exercise 2.7

Solve the following problems

1. In material design, a composite's properties are modeled by $M_t = CM_o$, where M_t represents tested properties, M_o initial properties, and C describes the composite's composition and processing. Given that $C = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ and $M_t = \begin{bmatrix} 1500 \\ 800 \end{bmatrix}$ after a series of tests, calculate the initial material properties.

2. Population growth of prey and predators is represented as $P_t = M\, P_o$, where P_t is the population at time t, P_o is the initial population, and M describes interactions between them. Given that $M = \begin{bmatrix} 1.2 & -0.5 \\ 0.3 & 0.1 \end{bmatrix}$ and $P_t = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$, calculate the initial populations.

3. Genetic traits in a species are modeled by $G_{parent} = M \times G_{offspring}$, where $G_{offspring}$ and G_{parent} are traits of offspring and parents, respectively, and M represents the inheritance matrix. Given that $M = \begin{bmatrix} 0 & -3 \\ 9 & 15 \end{bmatrix}$. and $G_{parent} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ calculate the genetic traits of the offspring represented by $G_{offspring.}$.

4. Temperature distribution in a composite rod is given by $T_t = kT_0$, with T_t as temperatures at different points, T_0 as initial temperatures, and k as the thermal conductivity matrix Given that $k = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.22 \end{bmatrix}$ and $T_t = \begin{bmatrix} 50 \\ 35 \end{bmatrix}$ calculate the initial temperatures.

5. In biomechanics, joint movement is described by S_t = J S_0, where S_t is the joint angles at various times, S_0 the initial angles, and J the matrix for joint kinematics.

   Given that $J = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.717 \end{bmatrix}$ and $S_t = \begin{bmatrix} 30 \\ 45 \end{bmatrix}$ calculate the initial joint angles.

6. A rectangular building with a length of 20 meters and a width of 30 meters is subjected to a force of 5000 Newton on its roof. The stress in the building can be calculated using the following matrix equation:

   Stress $= A \times B \times F$

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Where, $A=\begin{bmatrix}100 & 750\\ 500 & 250\end{bmatrix}$ and, $B=\begin{bmatrix}350\\ 100\end{bmatrix}$

(Matrix A represents building properties, B the applied force, and 'force' is its scalar magnitude.)

7. A bridge is designed with the following load-bearing matrix:

$A=\begin{bmatrix}75 & 200\\ 175 & 50\end{bmatrix}$.If the number of cars on the bridge is given by the matrix car $=\begin{bmatrix}25\\ 10\end{bmatrix}$ find the stress on the bridge. [Hint: use the formula, stress $= A\times cars$]

8. A suspension bridge is designed with the load-bearing matrix, $A=\begin{bmatrix}23 & 75\\ 65 & 15\end{bmatrix}$. If the numbers of vehicles in a bridge is given by the matrix, $B=\begin{bmatrix}x\\ y\end{bmatrix}$ and the stress in a bridge is given by a matrix, $S=\begin{bmatrix}90\\ 35\end{bmatrix}$ then find the number of cars in bridge

9. Express the system where twice the cars plus pedestrians equals 8, and four times the cars minus twice the pedestrians equals 10, in a matrix. Use matrix inversion to find the number of cars and pedestrians, and consider a related load-bearing matrix for a bridge. $A=\begin{bmatrix}50 & 100\\ 135 & 25\end{bmatrix}$, then find the stress on a bridge.

Review Exercise 2

1. Choose the correct answer in each of the following problems.

(i) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is

(a) An identity matrix w.r.t multiplication
(b) A column matrix
(c) An identity matrix w.r.t addition
(d) A null matrix

(ii) The matrix $\begin{bmatrix}4 & 0\\ 0 & -12\end{bmatrix}$ is

(a) A scalar matrix
(b) 2×3 matrix
(c) A diagonal matrix
(d) None of these

(iii) If $A=\begin{bmatrix}-1 & -2\\ 3 & 1\end{bmatrix}$, then $adjA$ is equal to

(a) $\begin{bmatrix}-1 & -2\\ 3 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\ -3 & -1\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}$

(iv) If $A=\begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}$, then A^{-1} equals

(a) $\begin{bmatrix}4 & 3\\ -3 & 2\end{bmatrix}$ (b) $\begin{bmatrix}4 & -3\\ -3 & 2\end{bmatrix}$

(c) $\begin{bmatrix}-2 & 3\\ 3 & -4\end{bmatrix}$ (d) $\begin{bmatrix}-4 & 3\\ 3 & -2\end{bmatrix}$

(v) For what value of d is the 2×2 matrix $\begin{bmatrix}5 & 1.5\\ 2 & d\end{bmatrix}$ NOT invertible?

(a) -0.6 (b) 0
(c) 0.6 (d) 3

(vi) Suppose A and B are 2×5 then $A+B$?

(a) 2×5 (b) 10×10
(c) 7×1 (d) 7×7

(vii) Inverse of $\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ is

(a) $\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & -2\\ 0 & -1\end{bmatrix}$

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(viii) The determinant of the matrix $\begin{bmatrix} 4 & -1 \\ -9 & 2 \end{bmatrix}$ is

(a) 17 **(b)** 1

(c) −1 **(d)** −17

2. Find x and y $\begin{bmatrix} x-1 & 4 \\ y+3 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -2 & -7 \end{bmatrix}$

3. Find the product if possible. $\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \begin{bmatrix} -6 & 5 & 8 \\ 0 & 4 & -1 \end{bmatrix}$

4. Find the inverse of the matrix $A = \begin{bmatrix} 6 & -3 \\ 5 & -2 \end{bmatrix}$

5. Solve the system $2x+5y=9$; $5x-2y=8$

6. Qasim sold 3 small and 14 large orange boxes for Rs.203, and Farzana sold 11 small and 11 large boxes for Rs.220. Calculate the price per small and large box.

7. You bought apples, bananas, and oranges. Apples cost Rs. 20 each with a tax rate of 5%, bananas cost Rs. 15 each with a tax rate of 4%, and oranges cost Rs. 10 each with a tax rate of 3%. Calculate the total cost for each type of fruit after adding tax.

8. For a week, you consumed 400 calories for breakfast, 600 calories for lunch, and 450 calories for dinner each day. Your daily calorie limit is 2000 calories, with breakfast accounting for 20%, lunch for 30%, and dinner for 25% of the total. Calculate your daily calorie intake.

9. In a classroom, there are three students, and each student has grades in four subjects: Math, English, Science, and History. Here are the grades (out of 100):

Student 1: [85, 90, 92, 88]

Student 2: [78, 86, 88, 90]

Student 3: [92, 94, 90, 87] The weight of each subject in the final grade is: Math (20%), English (25%), Science (30%), and History (25%). Calculate the final grades for each student.

SUMMARY

1. A matrix is a rectangular array of real numbers enclosed in brackets.
2. The rows of a matrix run horizontally, and the columns of a matrix run vertically.
3. A matrix with m rows and n columns has order $m \times n$ (read " m by n ").
4. Each number in a matrix is called an element or entry of the matrix.
5. Two matrices of the same order are equal if their corresponding elements are equal.
6. A matrix is said to be a row matrix if it has only one row.
7. A matrix is said to be a column matrix if it has only one column.
8. A matrix in which the number of rows is equal to the number of columns is called a square matrix.
9. A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix.
10. A matrix is said to be a zero matrix or null matrix if all its elements are zero.
11. If all the elements of a square matrix except the diagonal elements are zero, then the matrix is called a diagonal matrix.
12. A diagonal matrix, whose all the diagonal elements are equal, is called a scalar matrix.
13. A scalar matrix of order n in which each diagonal element is 1 (unity) is called an identity matrix of order n.
14. A matrix obtained by interchanging rows and columns of a matrix A is called the transpose of the matrix A and is denoted by A^t.
15. If a square matrix $A = A^t$, then A is called a symmetric matrix.

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16. A square matrix A is said to be skew-symmetric if $A^t = -A$.

17. $(A+B)^t = A^t + B^t, (A-B)^t = A^t - B^t$.

18. Two matrices are said to be conformable for addition/subtraction if they are of the same order.

19. The null matrix is the additive identity for the matrix addition.

20. If A is a matrix then $-A$ is the additive inverse of. A.

21. Two matrices A and B are said to be conformable for multiplication AB, only when the number of columns of matrix A is equal to the number of rows of matrix B.

$$A_{m\times p} \times B_{p\times n} = AB_{m\times n}$$

22. If A, B and C are three matrices then $A(BC) = (AB)C$. This property is called associative law of matrices w.r.t multiplication.

23. Commutative law of multiplication does not hold in matrices in general.

24. Let A, B and C be three matrices, then

(i) $A(B+C) = AB + AC$

(Left Distributive Law)

(ii) $(A+B)C = AC + BC$

(Right Distributive Law)

These are called distributive laws under multiplication over addition.

25. With every square matrix A, a unique real number is associated which is called the determinant of A denoted by $|A|$ or *det* A.

26. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

27. A square matrix A is called singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

28. The adjoint of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $adjA$ and defined as $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

29. If A and B are square matrices such that $AB = BA = I$ where I is the identity matrix. Then B is called the inverse of A, denoted as $B = A^{-1}$.

$$AA^{-1} = I = A^{-1}A$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

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