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Creativity

Geometry

Calculation

Grade

9

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A Textbook of

Mathematics

Grade 9

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for Grade 9
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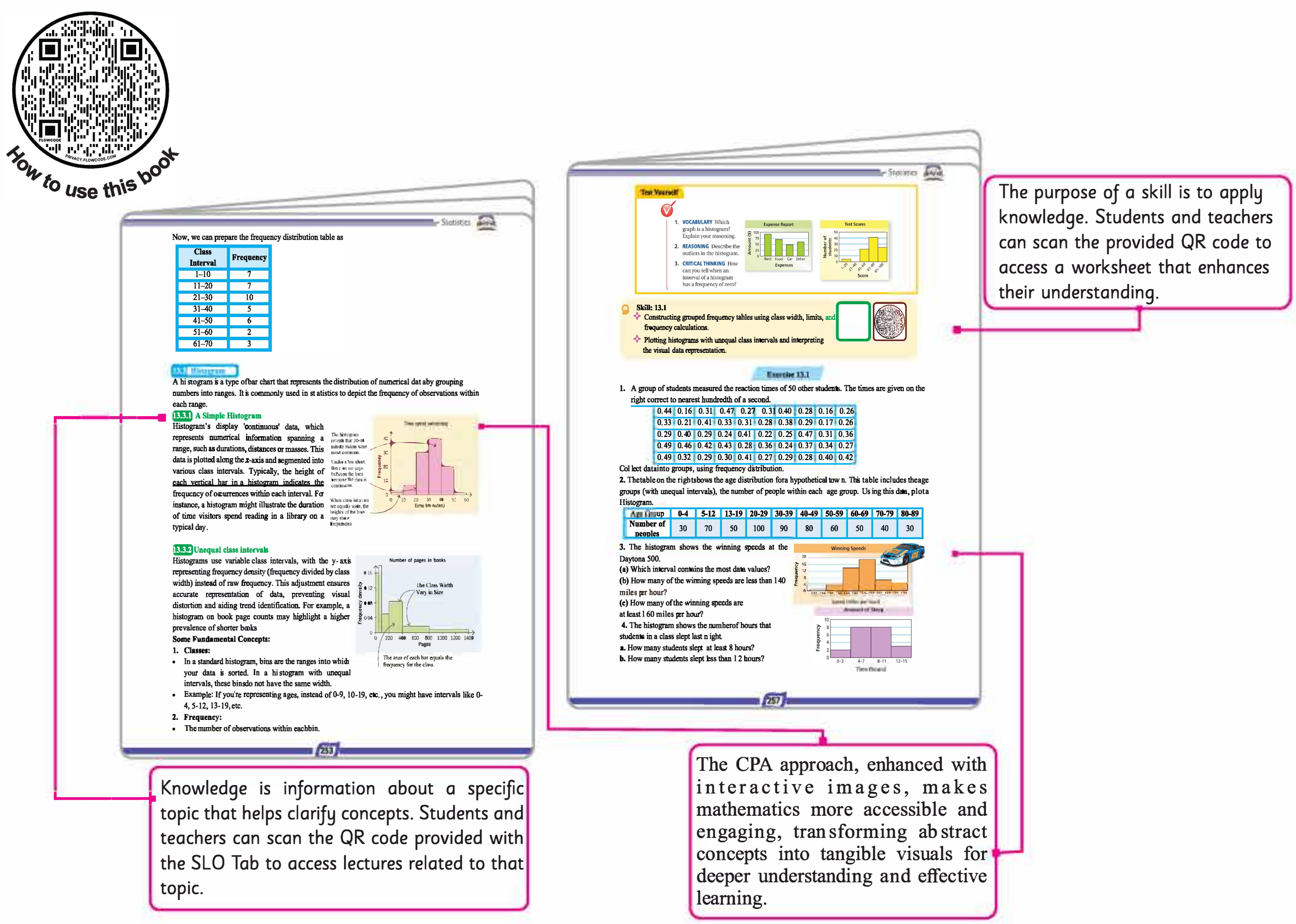
Preface

In this dynamic 9th-grade mathematics textbook, I embrace the evolving world of education by utilizing the CPA (Concrete, Pictorial, Abstract) Approach. This method, grounded in concrete examples, pictorial representations, and abstract concepts, caters to diverse learning styles, making mathematics accessible and engaging. Interactive images and real-life examples transform mathematical theories into vivid, relatable experiences, enhancing understanding and enjoyment.

The book encourages active learning through "Test Yourself" sections, classroom activities promoting collaboration and critical thinking, and insightful "Teacher's Footnotes" for effective content delivery. Rich in interactive color images, it offers a visually stimulating learning environment, breaking the monotony of traditional texts.

With a variety of examples, worksheets, and video lectures, the textbook provides comprehensive practice and learning opportunities. Additionally, simulations allow hands-on exploration of concepts, deepening understanding. This textbook is more than an educational tool; it's a journey designed to instigate a deep appreciation for mathematics, connecting the subject with the rhythm of the modern educational landscape.



SLO based Model Video lecture

Salient Features

Comprehensive Learning
Engage students with videos, simulations, and practical worksheets.

Structured Lesson Plan
Well-organized with clear objectives, PPTs, and a question bank.

Engaging Multimedia
Visual appeal through PPTs and interactive simulations.

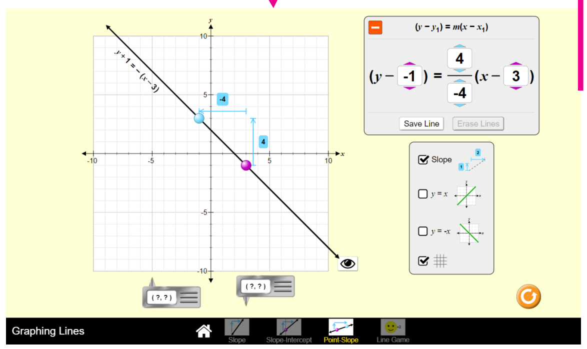
Assessment & Tracking
Diverse question bank and progress monitoring.

Adaptable & Accessible
Scalable and accessible, suitable for all learners.

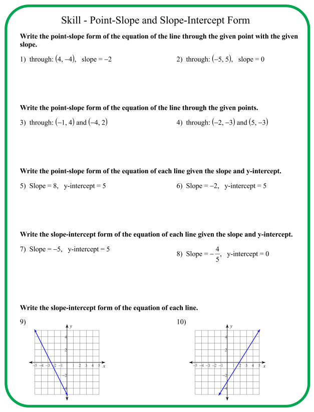
SLO No: M - 09 - B - 08

SLO statement: Derive Equation of a straight line in slope-intercept form & one point form

Simulation



Skills Sheet



Skill - Point-Slope and Slope-Intercept Form

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through: $(4, -4)$, slope $= -2$

2) through: $(-5, 5)$, slope $= 0$

Write the point-slope form of the equation of the line through the given points.

3) through: $(-1, 4)$ and $(-4, 2)$

4) through: $(-2, -3)$ and $(5, -3)$

Write the point-slope form of the equation of each line given the slope and y-intercept.

5) Slope $= 8$, y-intercept $= 5$

6) Slope $= -2$, y-intercept $= 5$

Write the slope-intercept form of the equation of each line given the slope and y-intercept.

7) Slope $= -5$, y-intercept $= 5$

8) Slope $= -\frac{4}{5}$, y-intercept $= 0$

Write the slope-intercept form of the equation of each line.

9)

10)

KNOWLEDGE

Student Learning Outcomes

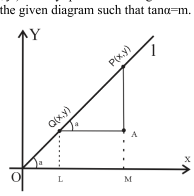
- Derive equation of a straight line in slope-intercept form, point-slope form, two-point form, intercepts form, symmetric form, normal form

7.7 Derivation of Standard Forms of Equations of Straight Lines

The derivation of standard forms for equations of straight lines is an important concept in geometry and algebra. These derivations explain how lines are represented in various mathematical forms, allowing for a deeper understanding of their properties and applications in the Cartesian coordinate system

7.7.1 Slope Intercept Form $y = mx + c$

Consider a straight-line L. Let $P(x, y)$ be any point on the given line L. Suppose that α is the inclination of the line L as shown in the given diagram such that $\tan\alpha = m$.



Take c as y-intercept of the straight line because it cuts the y-axis at the point Q(0, c), i.e., OQ= c = y-intercept. From point P draw PM perpendicular to the x-axis and from Q draw QR perpendicular to the PM. Now from the given diagram, consider the triangle ΔPQR, i.e., $m\angle RQP = \alpha$. By the definition of slope, we take

$\tan\alpha = \frac{PR}{QR} = \frac{PM-RM}{OM}$

$\Rightarrow \tan\alpha = \frac{PM-OQ}{OM}$

$\tan\alpha = \frac{y-c}{x}$

Now by the definition we can use 'm' instead of $\tan\alpha$ and we get

$m = \frac{y-c}{x}$

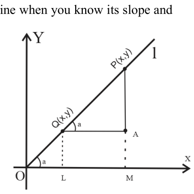
$mx = y - c$

$y = mx + c$

7.7.2 Point-Slope Formula

The point-slope formula is a method to determine the equation of a line when you know its slope and a point on it. The formula is:

$$y - y_1 = m(x - x_1)$$



To derive the point-slope formula, consider a line with slope 'm' and a known point (x_1, y_1). Let (x, y) be any other point on the line. The definition of slope is the ratio of the difference in y-coordinates to the difference in x-coordinates.

Therefore: $m = \frac{y-y_1}{(x-x_1)}$

Multiplying both sides with $x - x_1$

$(x - x_1) \times m = \frac{y-y_1}{(x-x_1)} \times (x - x_1)$

Content List

CHAPTER

1 Real Number System

Did you know that real numbers are super important for sending rockets into space and exploring other planets? Imagine we want to send a spacecraft to Mars. Scientists use real numbers to figure out how to get the spacecraft to go in the right direction and how fast it should travel to escape Earth's pull and not miss Mars. They also use these numbers to make sure the spacecraft can land on Mars exactly when and where they want it to, even though Mars is moving. This is like hitting a moving target from millions of miles away! Real numbers help scientists plan the spacecraft's path through space so it can safely reach Mars, orbit around it, or land on its surface. This planning is what makes it possible for us to send robots and even humans to explore space, showing just how cool and powerful math can be in helping us learn more about the universe.

Students' Learning Outcome

1. Explain, with examples, that civilizations throughout history have systematically studied living things (e.g., the history of numbers from Sumerians and its development to the present Arabic system.)
2. Describe the set of real numbers as a combination of rational and irrational numbers
3. Demonstrate and verify the properties of equality and inequality of real numbers
4. Apply laws of indices to simplify radical expressions
5. Apply concepts of rational numbers to real word problems (such as inventory (stock taking), temperature, banking, measures of gain and loss, sources of income and expenditure).

Pre & Post Requisite

Class 6
Chapter # 2
Integers

Class 7
Chapter # 1
Rational Number

Class 8
Chapter # 1
Real Numbers

Class 9
Chapter # 1
Real Number Systems

Class 10
Chapter # 1
Complex Numbers

Class 11
Chapter # 1
Complex Number

Knowledge

1. **Historical Development**
   The historical development of number systems through various civilizations.
2. **Classification of Numbers**
   The classification of numbers, including real, rational and irrational numbers.
3. **Properties of Addition and Multiplication**
   The properties of addition and multiplication for real numbers including closure, identity element, inverse element, commutativity, associativity and distributivity.
4. **Properties of Equality**
   The properties of equality for real numbers, including reflexive, symmetric and transitive properties etc.
5. **Properties of Inequality**
   The properties of inequality for real numbers, including transitive, trichotomy and additive, multiplicative, cancellation properties etc.
6. **Exponents and Radicals**
   The concepts of indices (exponents), radicals, radicands and surds.
7. **Laws of Exponents**
   The laws of indices and the interrelationship between radicals and exponents.
8. **Real Life Problems**
   The real-world applications of rational numbers, especially in situations related to banking, gains and losses, temperature conversions, inventory management etc..

Skills

1. Identifying and differentiating between different types of numbers (rational, irrational, real).
2. Applying the mathematical properties of addition and multiplication for real numbers.
3. Verifying the properties of equality and inequality for real numbers.
4. Applying the concept of indices and the laws of indices.
5. Understanding and applying the concept of radicals and radicands.
6. Rationalizing the denominators of radical expressions to eliminate radicals.
7. Transforming expressions between radical and exponent forms.
8. Simplifying expressions using the laws of exponents.
9. Applying mathematical concepts to solve real-world problems, specifically using rational numbers in contexts such as banking, gains and losses.

Introduction

Welcome to the fascinating world of mathematics! Within this chapter, we will embark on an exploration of the evolution of number systems, unravel the mysteries of real, rational and irrational numbers, uncover the foundational properties of addition and multiplication and journey through the complex realms of exponents, radicals and the practical integration of rational numbers in real-world scenarios. We invite you to join us on this engaging and enlightening journey of mathematical discovery!

Student Learning Outcomes

- Explain, with examples, that civilizations throughout history have systematically studied living things (e.g., the history of numbers from Sumerians and its development to the present Arabic system).

Knowledge 1.1 Mathematical Evolution Across Civilizations and Different Numeral System

1.1.1 Evolution of Mathematics

Throughout history, the integration of mathematical number systems with the study of living things has been a hallmark of various civilisations, reflecting a deep understanding of numbers about the natural world. Here are critical historical highlights

Early Civilizations: Egyptian and Babylonian number systems facilitated the recording and analysis of natural phenomena. For instance, the Egyptians applied geometry for land measurement and predicted Nile floods, which were essential for agriculture.

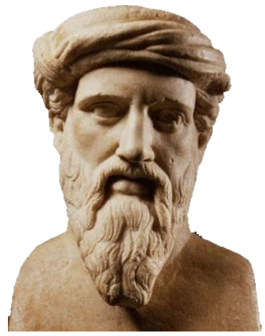
A clay disc featuring a sketch of a square, its diagonals, and markings that approximate the square root of 2, demonstrating its significance to the Babylonians

Hieroglyphs on the temple at ancient Ombos, near modern Kawm Umbu, Egypt

Greek Mathematics: Greek contributions, notably from Euclid and Pythagoras, were crucial in describing natural patterns. Pythagoras explored the relationship between numbers and musical harmonies, while Euclid's work in geometry provided foundational knowledge.

Euclid (325 BC–265 BC)

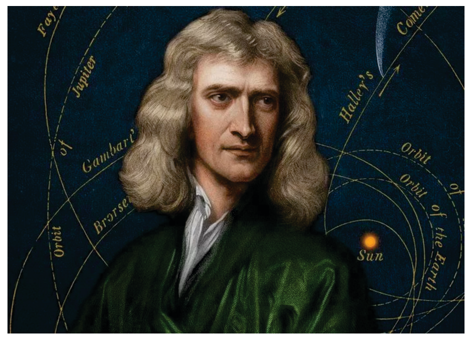
Indian and Chinese Mathematics: Introducing concepts like zero and negative numbers in India and China enriched the mathematical analysis of nature. Indian mathematicians advanced in astronomical calculations, while Chinese scholars developed methods for solving complex equations.



Pythagoras of Samos (570 BC–495 BC)

Islamic Golden Age: Scholars like Al-Khwarizmi advanced mathematical understanding, influencing the study of living organisms. His work in algebra and algorithmic techniques led to more systematic and precise scientific studies.

Renaissance Era: The development of calculus by figures like Newton and Leibniz revolutionized the understanding of natural phenomena. Newton's laws of motion and gravity unified the study of celestial and terrestrial bodies, offering a comprehensive mathematical framework for the natural world.



Isaac Newton (1643-1727)

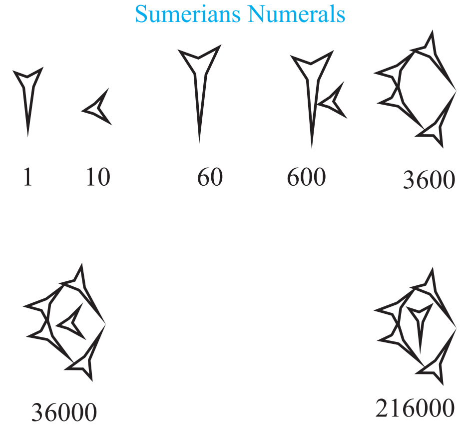
1.1.2 Different Numeral Systems Throughout History

Throughout history, civilizations have systematically studied living things, leading to the development of numeral systems for practical purposes like counting animals, managing households, and overseeing agricultural tasks. This drive for quantification resulted in diverse mathematical systems across cultures, from ancient Egyptian hieroglyphics for record-keeping to Roman numerals for commerce and the binary system for modern computing. These advancements underscore the integral role of mathematics in communication and understanding our world, highlighting the deep-rooted connection between our quest to quantify life and the evolution of mathematical practices. This narrative demonstrates how the systematic study of the natural world has been a constant endeavor, shaping the mathematical tools and systems we use today.

Sumerian Numerals (c. 4000-3000 BCE)

The early Sumerian civilization in Mesopotamia developed a base-60 (sexagesimal) numeral system, using cuneiform script for record-keeping, trade and astronomy.

Sumerians Numerals



Multiplication in Sumerians numerals

$3600 \times 10 = 36000$ $3600 \times 60 = 216000$

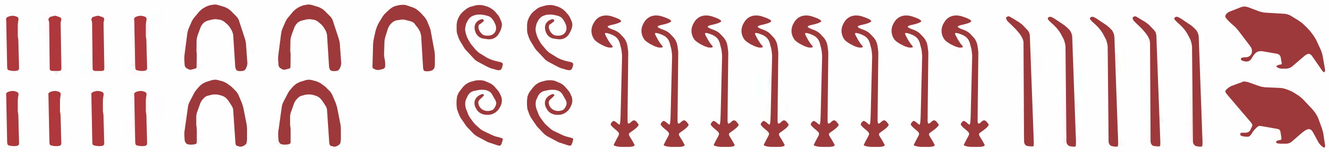
Egyptian Hieroglyphic Numerals (c. 3000-2000 BCE)

Ancient Egyptians employed a decimal system, with hieroglyphs representing numbers. They used basic symbols for 1-9 and unique symbols for powers of 10, primarily for counting and practical calculations.

1 10 100 1,000 10,000 100,000 1,000,000

Egyptian Hieroglyphic Numerals

Example of a Calculation in Egyptian Hieroglyphic numerals



8(1) + 5 (10) + 4 (100) + 8 (1,000) + 5 (10,000) + 2 (100,000) = 258458

Babylonian Numerals (c. 1800-1700 BCE)

Like the Sumerians, the Babylonians employed a base-60 (sexagesimal) numeral system. They used two symbols, one for 1 and another for 10. This system excelled at handling fractions and found applications in trade, astronomy and mathematics.

Greek Numerals (c. 800 BCE - 399 CE)

Ancient Greeks used letters for numbers, associating the first nine with 1-9, the following nine with tens and the subsequent nine with hundreds, serving various purposes in Greek text.

Roman Numerals (c. 1st millennium BCE - 16th century CE)

Roman numerals, including I for 1, V for 5, X for 10, L for 50, C for 100, D for 500 and M for 1000, were employed in trade and record-keeping but notably lacked representation for zero in mathematics.

Discovery of Zero

Zero originated in 5th-century South Asia as a dot, evolving into the '0' digit in the Arab world. It became part of the Hindu-Arabic numeral system, spreading to China and the Middle East. Fibonacci introduced zero to Europe around 1200 AD. The name 'zero' transformed linguistically from its South Asian origin to 'Sifr' in the Middle East, 'Zefero' in Italy and 'Zero' in English, reflecting a collective achievement shaped by various cultures over centuries

Indian Numerals (c. 5th century CE)

Indian civilization invented the decimal number system with digits 0 to 9 and positional notation, forming the basis for the modern numeral system.

Arabic Numerals (c. 9th century CE – present)

Derived from the Indian decimal number system, Arabic numerals, 0 to 9, with positional notation, revolutionised math, commerce and science in the Islamic Golden Age and became the global standard.

1.1.3 The Evolution and Impact of Arabic Numerals in Europe

Al-Khwarizmi and Al-Biruni's contributions refined the Hindu-Arabic numeral system, introducing the concept of zero and place values. Their work not only influenced Europe but also marked a historic shift from Roman numerals to Arabic numerals. Today, these numerals stand as a testament to the enduring legacy of Arabic characters and their profound impact on mathematics and science worldwide.

Babylonian	Greek	Roman	Indian	Arabic	Western
𒑊			०	٠	0
𒁹	α	I	१	١	1
𒈫	β	II	२	٢	2
𒐈	γ	III	३	٣	3
𒐉	δ	IV	४	٤	4
𒐊	ε	V	५	٥	5
𒐋	ϛ	VI	६	٦	6
𒐌	ζ	VII	७	٧	7
𒐍	η	VIII	८	٨	8
𒐎	θ	IX	९	٩	9
𒌋	ι	X	१०	١٠	10

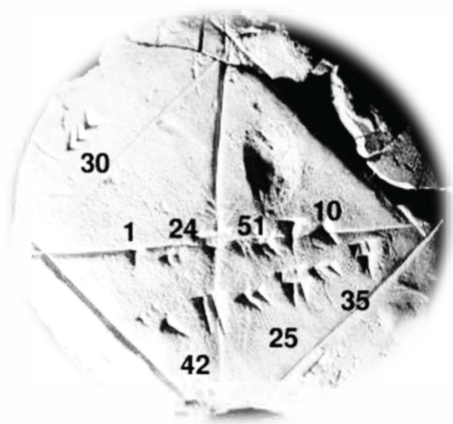
Comparison of Numeral Systems: Babylonian, Greek, Roman, Indian, Arabic, and Western Numerals

Interesting Information

Throughout history, different cultures have developed unique numeral systems to count, trade, and record information. From the ancient Egyptians' hieroglyphics to the Roman numerals and the binary system used in modern computers, these systems reflect the evolution of mathematics and communication. Let's explore the fascinating journey of numeral systems across civilizations and time periods

Activity

This clay tablet from the Babylonian Civilization measures the square root of 2 using the diagonal of a square. As depicted in the picture, the numbers 1, 24, 51 and 10 are inscribed along the diagonal line. These numbers belong to the sexagesimal system, based on a base 60. We will calculate the square root of 2 from this clay tablet and determine the difference between the value obtained through this ancient numeral system and that derived using a modern-day calculator.



To understand how this number approximates the square root of 2, let's break down the sexagesimal number:

- "1" is in the 'ones' place,
- "24" is in the 'sixtieths' place (like 'tenths' in decimal),
- "51" is in the 'three-thousand-six-hundredths' place (like 'hundredths' in decimal),
- "10" is in the 'two-hundred-and-sixteen-thousandths' place (like 'thousandths' in decimal).

So, the number 1245110 in sexagesimal translates to:

$$1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}$$

Converting this to decimal to see how it approximates the square root of 2:

$$1+\frac{24}{60}+\frac{51}{3600}+\frac{10}{216000}\approx 1.414213$$

The sexagesimal number 1245110, when converted to decimal, equals approximately 1.414213. The actual square root of 2 is approximately 1.414214. The difference between these two values is about 0.0000006, which is extremely small. This demonstrates that the ancient approximation of the square root of 2 in the sexagesimal system was remarkably accurate.

Knowledge 1.2 Introduction to Real Numbers

Student Learning Outcomes

- Describe the set of real numbers as combination of rational and irrational numbers.

Concreate-Pictorial-Abstract (CPA) Approach

Navigating the world of real numbers, we dissect it into rational and irrational components. This exploration sheds light on how these elements interact, offering a clear insight into the intricate landscape of mathematics

1.2.1 Rational Number

The collection of rational numbers, denoted by $\mathbb{Q}$, derives its name from 'ratio' and 'quotient'. All numbers in the form of $\frac{p}{q}$ where p and q are integers and q is not equal to zero, are called rational numbers, i.e., $\mathbb{Q}=\left\{x \mid x=\frac{p}{q};\, p,\ q\in\mathbb{Z}\wedge q\neq 0\right\}$. For example, –25 can be written as $-\frac{25}{1}$, here $p=-25$ and $q=1$.

1.2.2 Terminating Decimals

A terminating decimal refers to a decimal with a finite number of digits in its decimal part. For example, $\frac{1}{4}=0.25$ and $\frac{5}{8}=0.375$.

1.2.3 Non-terminating Recurring Decimals

A non-terminating recurring decimal is a decimal number that goes on indefinitely without ending and has a repeated sequence of digits after the decimal point. For example, $0.666\ldots, 1.34343434\ldots$

Note

Rational numbers may have equivalent forms (e.g., $\frac{1}{2}=\frac{2}{4}$) but we prefer the simplest form, where the numerator and denominator share no common divisors except 1.

These types of decimals can be written with a bar or the dots after the repeated digits, i.e., $0.\bar{6} = 0.66666\ldots$, $1.\overline{34} = 1.343434\ldots$
All non-terminating recurring decimals are rational, as they can be converted to a fraction through just a few simple steps.

Let's consider an example to illustrate this concept.

Example 1.1 Convert the non-terminating recurring decimal $0.\overline{12}$ into a fraction.

Solution

Step 1: Let $x = 0.121212\ldots$ .... (i)

Step 2: Multiply by 100 to shift the decimal two places right (for two recurring digits).

$$100x = 12.121212\ldots \quad \text{.........} (ii)$$

Step 3: Subtract (i) from (ii) to eliminate the recurring part.

$$100x - x = 12.121212\ldots - 0.121212\ldots$$
$$99x = 12$$

Step 4: Divide both sides by 99 to solve for x.

$$x = \frac{12}{99}$$

Step 5: Simplify the fraction.

$$x = \frac{4}{33}$$

So, the non-terminating recurring decimal 0.121212... is equal to the fraction $\frac{4}{33}$.

Hence, we can say that, "Every non-terminating, recurring decimal can always be expressed as a rational fraction."

Teacher's Guidelines

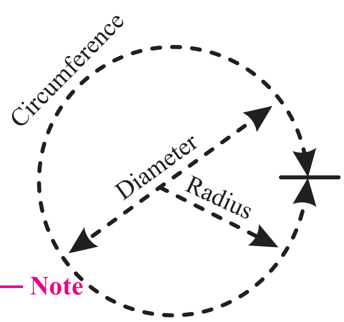
- **Bring a thermometer:** Show a digital or traditional thermometer.
- **Explain real numbers**: Discuss how real numbers include integers, fractions, and decimals.
- **Demonstrate temperature readings**: Show temperatures as whole numbers (e.g. 25°C), fractions, $25\frac{1}{2}$ °C) or decimals (e.g., 25 .5°C)
- **Relate to real-life:** Discuss how different temperatures are relevant in daily life.
- **Engage with examples**: Ask students for temperature examples from their experiences, identifying them as whole numbers, fractions, or decimals.

Interesting Information

The groundbreaking discovery of irrational numbers in the 5th century BCE, notably the square root of 2 derived from the hypotenuse of a unit isosceles right triangle, posed a significant challenge to the prevailing belief among Pythagoreans that all numbers were rational. Hippasus of Metapontum, a member of the Pythagorean school, is often credited with this revelation. However, the unveiling of this truth led to controversy within the Pythagorean community, and legend has it that Hippasus faced dire consequences, possibly meeting a tragic end for challenging established mathematical doctrine.

Activity

Take a string make a circle of it. find the diameter using a meter rule. Divide the circumference (length of string) with diameter, you will get a non-terminating, non-recurring decimal called 'Pi' (π).
$\pi = 3.14159265358979323846\ldots$



1.2.4 Irrational Numbers

The numbers which cannot be expressed as quotient of integers are called irrational numbers. The set of irrational numbers is denoted by $\mathbb{Q}'$,

$$\mathbb{Q}' = \left\{ x | x \neq \frac{p}{q};\ p, q \in \mathbb{Z} \wedge q \neq 0 \right\}$$

For example, the numbers $\sqrt{2}$,$\sqrt{3}$, $\sqrt{5}$ are irrational numbers.

1.2.5 Non-terminating, Non-recurring Decimal

A non-terminating, non-recurring decimal features an infinite repetition of random digits after the decimal point. For example, $0.3457812\ldots, 4.5675\ldots$.. In non-recurring decimal, digits are repeated randomly hence they can't be converted into fractions, for example, $\sqrt{2} = 1.414213\ldots$, $\pi = 3.1415926535\ldots$, $e = 2.71828182845904\ldots$.

Example 1.2 Write three irrational numbers between 4 and 4.5

Solution:

We have to find some numbers in between 4 and 4.5 that are non-terminating and non-recurring $4.124534\ldots, 4.45321213\ldots$, $4.2586268268\ldots$. The square roots $\sqrt{16.2}$, $\sqrt{18}, \sqrt{17}$ are also non-recurring and lies between 4 and 4.5.

Note

- Square root of all prime numbers is irrational.
- Square root of any rational number which is not a perfect square is always irrational.
- Rational and irrational number are two disjoint sets (they have nothing in common).
- Addition/Subtraction of irrational number with any number can be rational or irrational.
- **Multiplication:**

Irrational × Rational: The result will be irrational. For example, $\sqrt{2} \times 3 = 3\sqrt{2}$, [Except for multiplication with 0, the result is 0 (Rational number)].

Irrational × Irrational: The result can be rational or irrational. For example, $\sqrt{2} \times \sqrt{2} = 2$, which is rational, but, $\pi \times \sqrt{2} = \pi\sqrt{2}$ which is irrational.

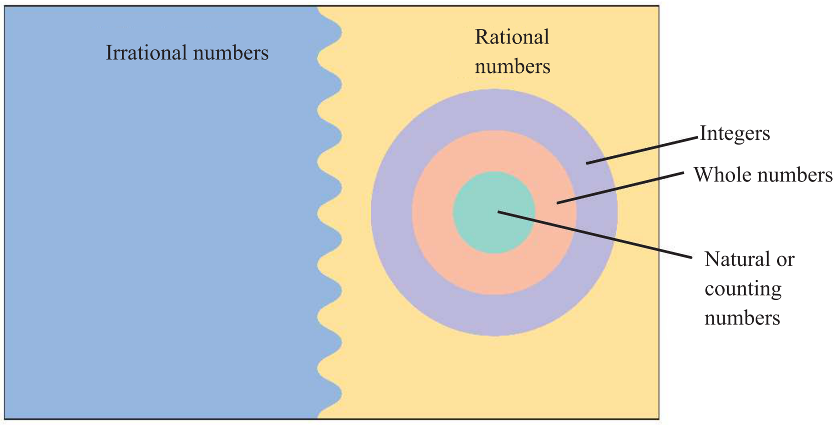
- **Division:**

Irrational ÷ Rational: The result will be irrational. For example, $\frac{\sqrt{5}}{2}$, [Except for division with 0, the result is undefined].

Irrational ÷ Irrational: The result can be rational or irrational. For example, $\frac{\sqrt{8}}{\sqrt{2}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2,$ which is rational, but $\frac{\pi}{\sqrt{2}}$ is irrational.

1.2.6 Real Numbers

Considering the various types of numbers, we can conclude that the numbers we use in daily life are either rational or irrational. These numbers, which can be plotted on a number line as you have studied in your previous classes, are known as real numbers.



Real numbers

The relationship of various types of numbers.

Skill 1.1
Identifying and differentiating between different types of numbers (rational, irrational, real).

Real numbers can be defined as the union of rational and irrational numbers, i.e., $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

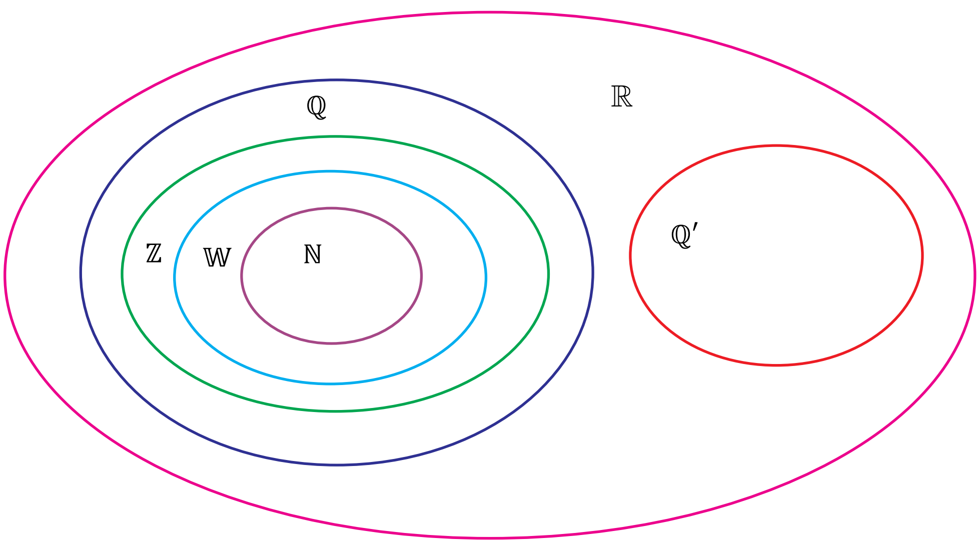
Exercise 1.1

1. Identify the following and place them in the circles accordingly.

(If a number falls in more than one group, placed it in every appropriate group).

i. $\sqrt{3}$ ii. $2+\sqrt{5}$ iii. -876 iv. $\frac{9}{8}$ v. $0.176767\ldots$ vi. $2.718281\ldots$ vii. π viii. 71

ix. $\frac{22}{7}$ x. 0.8652 xi. 12 xii. 4π xiii. $\sqrt{64}$ xiv. $\sqrt{1\frac{7}{9}}$ xv. $-6\sqrt{2}$ xvi. $\frac{4\pi}{3\pi}$



2. **Convert the following rational fractions into decimal form.**

i. $\frac{17}{25}$ ii. $\frac{3}{8}$ iii. $\frac{11}{25}$ iv. $\frac{2}{7}$

3. **Convert each of these decimals into its simplest rational fraction form.**

i. 6.7 ii. 0.056 iii. 0.276 iv. 0.7428

4. **Give any three random rational numbers between the following,**

i. $\frac{3}{4}$ and $\frac{5}{9}$ ii. 0 and 1 iii. 3.15 and 4.5

5. **Give a rational number between the following**, (Hint**:** Find the average of the two numbers)

i. $\frac{1}{3}$ and $\frac{1}{4}$ ii. $\frac{5}{6}$ and $\frac{11}{12}$ iii. -5 and -4

6. **Give any two irrational number between the following,**

i. 3 and 4 ii. 10 and 11 iii. 19 and 20

7. **Convert the following recurring decimals into rational numbers in the form of $\frac{p}{q}$, where p and q are integers and $q \neq 0$.**

i. $0.\overline{5}$ ii. $0.\overline{67}$ iii. $1.\overline{34}$

Knowledge 1.3 Properties of Real Numbers

Student Learning Outcomes

- Demonstrate and verfiy the properties of equality and inequality of real numbers.

In this section, we will cover the properties of real numbers, a fundamental aspect of mathematical systems. The real number system includes real numbers and operations like addition, subtraction, multiplication and division, and the corresponding properties are known as the properties of real numbers.

Operation	Property	Description/Example
Closure Property		Closure is derived from word closed, if the participants and their resultant belongs to the same set then they are said to be closed.
Addition	$a+b \in \mathbb{R} \quad \forall\, a,b \in \mathbb{R}$	The sum of two real numbers is always a real number. **Example:** $1+5=6$ is a real number.
Multiplication	$a \times b \in \mathbb{R} \quad \forall\, a,b \in \mathbb{R}$	The product of two real numbers is always a real number. **Example:** $7\times 3=21$ is a real number.
Associative Property		Associativity is to link three participants in any order but resultant remains the same.

Addition	$(a+b)+c=a+(b+c)$ $\forall\, a,b,c \in \mathbb{R}$	When three real numbers are added, it makes no difference which two are added first. **Example:** $(1+5)+8=1+(5+8)$ $6+8=1+13$ $14=14$
Multiplication	$(a\times b)\times c= a\times(b\times c)$ $\forall\, a,b,c \in \mathbb{R}$	When three real numbers are multiplied, it makes no difference which two are multiplied first. **Example:** $4(3\times 5)=(4\times 3)5$ $4\times 15=12\times 5$ $60=60$
Identity Element		A unique element in a set that, when added to or multiplied by any other element, does not change its value, is referred to as an 'identity element'
Addition	$a+0=a$ $\forall\, a \in \mathbb{R}$	The sum of zero and a real number equals the number itself. **Example:** $0+5=5$ **Note:** 0 is an additive identity.
Multiplication	$a\times 1=a$ $\forall\, a \in \mathbb{R}$	The product of one and a real number equals the number itself. **Example:** $1\times 5=5$ **Note:** 1 is a multiplicative identity.
Inverse Element		For every real number, there exists a unique real number such that when they are operated upon, they yield the identity. $a+a'=\text{Identity}$ $a\times a'=\text{Identity}$ a and a' are said to be inverse of each other. When they are added, the result is 0 and when they are multiplied, the result is 1.
Addition	$a+(-a)=0$ $\forall\, a \in \mathbb{R}$	The sum of a real number and its opposite is zero. **Example:** $5+(-5)=0$ (Additive identity) So, 5 and -5 are additive inverses of each other.

The product of a real number and its reciprocal is 1.

Multiplication	$a\times\frac{1}{a}=1$ $\forall\, a\in\mathbb{R}$	**Example:** $5\times\frac{1}{5}=1$ (Multiplicative identity) So, 5 and $\frac{1}{5}$ are multiplicative inverses of each other.
Commutative Property		If changing the order of operation in two elements does not alter the resultant, they are said to be commutative.
Addition	$a+b=b+a$ $\forall\, a,b\in\mathbb{R}$	Two real numbers can be added in either order. **Example:** $1+5=5+1$ $6=6$
Multiplication	$a\times b=b\times a$ $\forall\, a,b\in\mathbb{R}$	Two real numbers can be multiplied in either order. **Example** $-2\times 8=8\times -2$ $-16=-16$
Distributive Property		When there are three elements and two operations, one operation can be distributed to the other two.
Multiplication distributes over addition. $\forall\, a,b,c\in\mathbb{R}$ $a(b+c)=ab+ac$ (Left distributive) $(a+b)c=ac+bc$ (Right distributive)		**Example:** $2(3+6)=2\times 3+2\times 6$ $2(9)=6+12$ $18=18$
Multiplication distributes over subtraction. $\forall\, a,b,c\in\mathbb{R}$ $a(b-c)=ab-ac$ (Left distributive) $(a-b)c=ac-bc$ (Right distributive)		**Example:** $(3-6)\times 2=3\times 2-6\times 2$ $-3\times 2=6-12$ $-6=-6$

Note

Subtraction and division are derived from addition and multiplication. Subtraction is $a-b=a+(-b)$ division is $a\div b=a\times\left(\frac{1}{b}\right)$.
However, division by 0 is undefined. Subtraction and division do not possess commutative and associative properties.

Let's solve some examples to understand the properties of real numbers and how they're used.

Example 1.3

Identify the properties of real numbers that correspond to each statement. (Note: a and b represent real numbers.)

i. $9\times 5=5\times 9$ **ii.** $4(a+3)=4(a)+4(3)$ **iii.** $6\left(\frac{1}{6}\right)=1$

iv. $-3+(2+b)=(-3+2)+b$ **v.** $(b+8)+0=b+8$

Solution:

i. This statement is justified by the "commutative property of multiplication".
ii. This statement is justified by the "left distributive property of multiplication over addition".
iii. This statement is justified by the "multiplicative inverse property".
iv. This statement is justified by the "associative property of addition".
v. This statement is justified by the "additive identity property".

Example 1.4

Complete each statement using the specified property of real numbers.

i. Multiplicative identity property: $(4a)1=$ _________

ii. Associative property of addition: $(a+9)+1=$ _________

iii. Additive inverse property: $0=5c+$ _________

iv. Distributive property: $4\times b+4\times 5=$ _________

Solution:

i. $(4a)1=4a$ **ii.** $(a+9)+1=a+(9+1)$

iii. $0=5c+(-5c)$ **iv.** $4\times b+4\times 5=4(b+5)$

1.3.1 Properties of Equality of Real Numbers

In this section we will discover the key rules that guide equality among real numbers, unraveling the foundations of mathematical relationships with clarity and simplicity

Property	Notation & Example	Description
1. **Reflexive Property**	$\forall a\in \mathbb{R}, a=a$ **Example:** $2=2, x=x$	Every value equals itself, a fundamental concept in mathematical proofs.
2. **Symmetric Property**	$\forall a, b\in \mathbb{R}\ a=b\Rightarrow b=a$ **Example:** $2=x\Rightarrow x=2$	Equal values maintain their equality even when the order is reversed, making it essential for equations and relationships.

3. Transitive Property	$\forall\ a, b, c \in \mathbb{R}, a = b$ and $b = c \Rightarrow a = c$ **Example:** $x+3=8;\ 8=2y \Rightarrow x+3=2y$	Links equalities, facilitating problem-solving.
4. Substitution Property	$\forall\ a, b, c \in \mathbb{R}, a = b$ and $a = c \Rightarrow b = c$ **Example:** In $x+2=10$, we know that $x=8$, we can use the substitution property to replace 'x' with '8' resulting in $8+2=10$.	Allows replacing variables with equivalent values without altering the truth of an equation.
5. Additive Property	$\forall\ a, b, c \in \mathbb{R}$ if $a=b$, then $a+c=b+c$ **Example:** In $x-5=10$, we can add 5 to both sides: $x-5+5=10+5$ $\Rightarrow\ x=15$	Adding the same number to both sides of an equation preserves its truth.
6. Multiplication Property	$\forall\ a, b, c \in \mathbb{R}$ if $a=b$, then $a \times c = b \times c$ **Example:** In $2x=10$, we can multiply both sides by $\frac{1}{2}$. $2x \times \frac{1}{2} = 10 \times \frac{1}{2}$ $\Rightarrow\ x=5$	Multiplying both sides of an equation by the same non-zero number maintains its validity.
7. Cancellation Property for Addition	$\forall\ a, b, c \in \mathbb{R}$ if $a+c=b+c$ then $a=b$ **Example:** $2+y=8$ Since "y" is being added to "2" we can remove "2" from the left side without changing the equation's truth: $2+y=2+6 \Rightarrow y=6$	Eliminate common terms from both sides to simplify equations and solve for unknowns.
8. Cancellation Property for Multiplication	$\forall\ a, b, c \in \mathbb{R}$ if $a \times c = b \times c$ then $a=b$ **Example:** $3x=15$ We can remove "3" from the left side of the equation without changing its truth. So, we have, $3(x)=(3)5$ $\Rightarrow\ x=5$	Dividing both sides of a multiplication equation by the same non-zero number preserves its validity.

Let's solve some examples using the properties of equality of real numbers:

Example 1.5 Solve the equation: $3(y-4)=15$

Solution:

Step 1: Distribute '3' on the left side using the multiplication property of equality: $3y-12=15$

Step 2: To isolate the variable term, use the addition property of equality by adding 12 to both sides:

$$3y-12+12=15+12$$

$$3y=27$$

Step 3: Solve for y, by dividing both sides by 3,

$$\frac{3y}{3}=\frac{27}{3}$$

So, the solution to the equation is $y=9$.

1.3.2 Properties of Inequality of Real Numbers

Inequality is a mathematical relationship between two values that indicates one value is greater than, less than, or not equal to the other value. It is denoted by symbols such as "<" (less than), ">" (greater than), "≤" (less than or equal to), "≥" (greater than or equal to), or "≠" (not equal to).

Property	Notation and Example	Description
1. Trichotomy Property	$\forall\ a, b \in \mathbb{R} \Rightarrow a<b$ or $a=b$ or $a>b$ **Example:** Let's consider "a" = 3 and "b" = 7. $a<b$: In this case, $3<7$ because 3 is strictly less than 7. $a=b$: Here, 3 is not equal to 7, so this condition does not apply. $a>b$: Since 3 is not greater than 7, this condition does not apply either.	For any two real numbers "a" and "b" exactly one of the following three relationships holds: "a" is greater than "b" or "a" is equal to "b," or "a" is less than "b."
2. Transitive Property	$\forall\ a, b, c \in \mathbb{R}$, **(i)** $a<b$ and $b<c \Rightarrow a<c$ **(ii)** $a>b$ and $b>c \Rightarrow a>c$ **Example:** Let us consider, $a=5; b=10; c=15$ We have the following inequalities: $a<b$ $(5<10)$ and $b<c$ $(10<15)$ According to the transitive property: If $a<b$ and $b<c$, then $a<c$ $(5<15)$.	If a relation holds between a first and second element and also between the second and third element, it automatically holds between the first and third element without the need for direct comparison.

3.Additive Property	$\forall\ a, b, c \in \mathbb{R}$ **(i)** $a < b \Rightarrow a + c < b + c$ **(ii)** $a > b \Rightarrow a + c > b + c$ **Example:** Consider $3<7$ By adding 2 to both sides: $3+2<7+2$ This simplifies to: $5<9$ The inequality $5<9$ remains true.	Adding or subtracting the same value from both sides of an inequality does not change its validity, facilitating mathematical problem-solving.
4.Multiplication Property	**a.** $\forall\ a, b, c \in \mathbb{R}$ and $c > 0$ (Positive) **(i)** If $a > b \Rightarrow ac > bc$ **(ii)** If $a < b \Rightarrow ac < bc$ **b.** $\forall\ a, b, c \in \mathbb{R}$ and $c < 0$ (Negative) **(i)** If $a > b \Rightarrow ac < bc$ **(ii)** If $a < b \Rightarrow ac > bc$ **Example:** Consider $3<7$ Multiply both sides by -2, $3(-2)>7(-2)$ $-6>-14$ The inequality $-6>-14$ is true, as -6 is greater than -14.	Multiplying both sides of an inequality by a positive number preserves the direction of the inequality, while multiplying by a negative number reverses it.
5.Multiplicative Inverse Property (Reciprocal of an Inequality)	$\forall\ a, b \in \mathbb{R}$ where $a \neq 0$, $b \neq 0$ **(i)** $a<b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$ (when a and b both are positive or negative) **(ii)** $a > b \Leftrightarrow \frac{1}{a} < \frac{1}{b}$ (when a and b have opposite signs) **Example:** If we have $3<6$, and we reciprocate both sides, we get $\frac{1}{3} > \frac{1}{6}$ If we have $-4<2$, and we reciprocate both sides, we get $\frac{1}{-4} < \frac{1}{2}$	Taking the reciprocal of both sides of an inequality reverses the direction if the values have the same sign, and maintains the direction if they have opposite signs. Avoid division by zero, as it leads to undefined results.

Note

Anti-reflexive Property: $\forall\, a \in \mathbb{R},\ a \nless a$ and $a \ngtr a$

Anti-symmetric Property:

$\forall\ a, b \in \mathbb{R}$ if $a < b$, then $b \nless a$. If $a > b$, then $b \ngtr a$.

Let's consider a few examples to explore and apply the properties of inequality in solving mathematical problems.

Example 1.6

Identify the property used in each step of the following inequality simplification:

$$3x+5<2x+8$$

Solution:

Step 1: Subtract $2x$ from both sides:

$$3x+5-2x<2x+8-2x$$

Step 2: Simplify both sides: $x+5<8$

Step 3: Subtract 5 from both sides:

$$x+5-5<8-5$$

Step 4: Simplify both sides: $x<3$

In this example, the addition and subtraction property were used in steps 1 and 3 to manipulate the inequality.

Example 1.7

Use the 'Transitive Property' to compare the following inequalities:

$$2x<8 \text{ and } 8<3x$$

Solution:

By the 'Transitive Property,' if $2x<8$ and $8<3x$, then we can directly compare $2x$ and $3x$:

$$2x<8<3x$$

$$2x<3x$$

Remember to carefully identify and apply the correct property in each step when solving inequalities. Practicing with more examples will help solidify your understanding of these properties.

Real World Application

The properties of real numbers are vital in mathematics and practical applications. They enable solving equations, simplifying expressions, and making decisions in areas like calculus, statistics, economics, and engineering, being fundamental for mathematical reasoning and problem-solving in various disciplines.

Skill 1.2

- Applying addition and multiplication properties for real numbers.
- Verifying equality and inequality properties for real numbers.

Exercise 1.2

1. Identify the property of real numbers used in following expressions.

i. $x+9=9+x$ ii. $2(x+3)=2x+6$ iii. $(x+y)+3=x+(y+3)$

iv. $(5y)\times(1)=5y$ v. $(xy)z=x(yz)$ vi. $(x+5)(7+x)=(x+5)\times(7)+(x+5)\times(x)$

vii. $(y+2)+(-y-2)=0$

2. Match the number sentences in column A with the corresponding properties of equality or inequality for real numbers in column B.

	Column A	Column B
i.	If $8+2<14$ and $14<20$ then $8+2<20$	Addition property of equality
ii.	If $(m-n)<(p+q)$ and $r>0$, then $(m-n)r<(p+q)r$	Multiplication property of equality
iii.	If $m=n$ then $m+p=n+p$	Multiplication Property of inequality
iv.	If $q+r=15$, then $15=q+r$	Transitive Property of inequality
v.	If $15y=75$ then $3y=15$	Symmetric Property

3. Fill in the following blanks by stating the properties of real numbers.

$3x+3(y-x)$

a) $=3x+3y-3x$____________ b) $=3x-3x+3y$____________

c) $=0+3y$____________ d) $=3y$____________

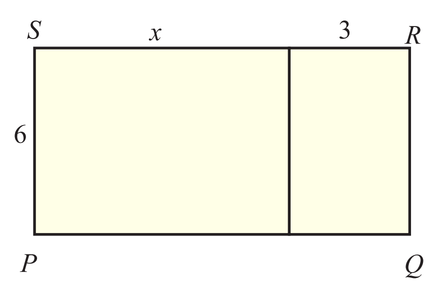
4. The area of rectangle $PQRS$ is $6(x+3)$, can be expressed as the sum of the areas of the two smaller rectangles, $6x+6(3)$. The fact that $6(x+3)=6x+6(3)$ illustrates which property?

a) The Distributive Property.

b) The Associative Property of Addition.

c) The Commutative Property of Addition.

d) The Transitive Property.



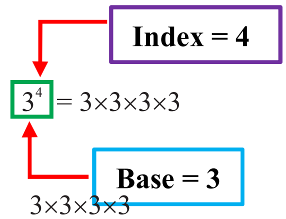
Knowledge 1.4 Indices

The word ‘index’ (plural indices) has many meanings in real life including a list of names, the index for a book and a price index, but the focus in this chapter is, of course, related to numbers.

Index (Exponent) notation is a shorthand way of writing numbers. For example, $3\times3\times3\times3$ can be written as 3^4.The notation 3^4 is called index or exponential (Power) notation. A power or an index, is used to write a product of numbers very compactly.
To simplify any exponent form, one should write it in its basic constituents.

Student Learning Outcomes

- Apply laws of indices to simplify radical expressions.



Example:

i. $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\left(2^3\right)^{\frac{1}{3}}\right)^2 = 2^2 = 4$

ii. $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\left(8^{\frac{1}{3}}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$

1.4.1 Laws of Exponents

Exponents are a fundamental concept in mathematics that allows us to express repeated multiplication in a concise way. While dealing with exponents, certain rules or "laws" have been established to simplify and streamline computations. These laws provide a framework for working with expressions involving powers or exponents.

(a) Product of Exponents (Law No. 1)

If you multiply two powers with the same base, you can add the exponents, i.e.,

$$a^m \times a^n = a^{m+n}$$

(b) Quotient of Exponents (Law No. 2)

If you divide two powers with the same base, you can subtract the exponents, i.e.,

$$\frac{a^m}{a^n} = a^{m-n}$$

(c) Power of an Exponent (Law No. 3)

If you raise a power to a power, you multiply the exponents.

$$\left(a^n\right)^m = a^{nm}$$

Example 1.8

Simplify the expression:

$$\frac{\left(x^3\right)^4 \times \left(x^2\right)^5}{x^7}$$

Solution:

First, apply the power rule,

$$\left(x^3\right)^4 = x^{3\times 4} = x^{12} \text{ and } \left(x^2\right)^5 = x^{2\times 5} = x^{10}$$

Now the expression is $\dfrac{\left(x^{12} \times x^{10}\right)}{x^7}$.

Next, apply the product of powers rule to the numerator,

$$x^{12} \times x^{10} = x^{12+10} = x^{22}$$

So, our expression is now,

$$\frac{x^{22}}{x^7}$$

Finally, we use the quotient of powers rule:

$$\frac{x^{22}}{x^7} = x^{22-7} = x^{15}$$

Therefore, $\dfrac{\left(x^3\right)^4 \times \left(x^2\right)^5}{x^7} = x^{15}$

Test Yourself

Convert the following areas using exponent laws:

a) Find the number of square meters in 1 square kilometer (1 km^2).

b) Find the number of square meters in one million square centimeters ($10^6 cm^2$).

c) Find the number of square millimeters in one million square centimeters ($10^6 cm^2$).

d) Find the number of square centimeters in 1 square kilometer (1 km^2).

(d) Zero Power Rule (Law No. 4)

Any non-zero number to the power of 0 is 1, i.e.,

$$a^0 = 1 \text{ (where } a \neq 0)$$

We know that when dividing exponents with the same base, we subtract the powers. So, if we have something like 2^3 divided by 2^3, using the rule, we subtract the exponents to get,

$$2^0 = 2^{3-3} = \frac{2^3}{2^3} = 1$$

(e) Negative Exponent Rule (Law No. 5)

A negative exponent means to divide 1 by the base raised to the power of the absolute value of the exponent, i.e.,

$$a^{-n} = \frac{1}{a^n}$$

Example 1.9

Simplify $\dfrac{\left(3x^2\right)^{-4}}{\left(-2y^{-1}z^2\right)^{-2}}$

Solution:

$$\frac{\left(3x^2\right)^{-4}}{\left(-2y^{-1}z^2\right)^{-2}} = \frac{3^{-4}(x^2)^{-4}}{(-2)^{-2}\left(y^{-1}\right)^{-2}\left(z^2\right)^{-2}}$$

$$= \frac{3^{-4}x^{-8}}{(-2)^{-2}y^2z^{-4}} = \frac{(-2)^2z^4}{3^4x^8y^2} = \frac{4z^4}{81x^8y^2}$$

Test Yourself

1. Why do you add the exponents to simplify $3^2 \times 3^4$, but multiply the exponents to simplify the expression $(3^2)^4$?

2. a) What is the difference between a quotient of powers and a power of a quotient?

b) What is the difference between a product of powers and a power of a product?

(f) Distributive Law of Exponents over Multiplication (Law No. 6)

It states that when raising a product to a power, you can distribute the exponent to each factor, i.e.,

$$(ab)^n \Leftrightarrow a^n \times b^n$$

Example 1.10

Simplify $16x^4y^8z^{12}$

Solution:

$$16x^4y^8z^{12} = \left(2^4x^4\left(y^2\right)^4\left(z^3\right)^4\right) = \left(2xy^2z^3\right)^4$$

(g) Distributive Law of Exponents over Division (Law No. 7)

It states that when raising a quotient to a power, you can distribute the exponent to both the numerator and the denominator, i.e.,

$$\left(\frac{a}{b}\right)^n \Leftrightarrow \frac{a^n}{b^n}$$

Note

1. If the bases a and b are negative then powers m and n must be integers for laws 1-3 and 5-6 to be always valid.
2. Laws 4 is also true if base a is negative.

Example 1.11

Simplify $\left(\dfrac{-x^3}{y}\right)^2$

Solution:

$$\left(\frac{-x^3}{y}\right)^2 = \frac{\left(-x^3\right)^2}{y^2} = \frac{x^6}{y^2}$$

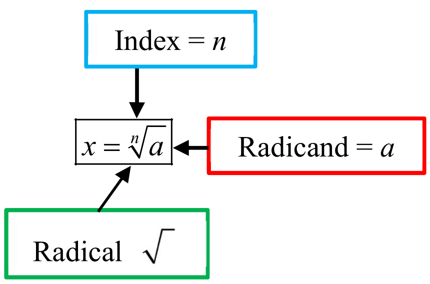
Knowledge 1.5 Radicals

A **radical** is a symbol that represents the root of a number. The most commonly used radical is the square root symbol $\left(\sqrt{\ }\right)$, but other roots exist, such as the cube root $\left(\sqrt[3]{\ }\right)$, fourth root $\left(\sqrt[4]{\ }\right)$ etc. The **radicand** is the number or expression underneath the radical.

For example, in the expression $\sqrt{16}$, the radical is the square root symbol $\left(\sqrt{\ }\right)$ and the radicand is 16. Index is the power of radical which is 2 in this case.

1.5.1 Surd

A surd is a specific type of radical expression. It refers to a root of a number that cannot be simplified to a whole number. For example, $\sqrt{2}$ is a surd because the square root of 2 does not simplify to a whole number. However, $\sqrt{4}$ is not a surd, because it simplifies to the whole number 2.



Hence, the radical $\sqrt[n]{a}$ is a surd if, a is rational such that the result $\sqrt[n]{a}$ is irrational.

e.g., $\sqrt{3}$, $\frac{\sqrt{2}}{5}$, $\sqrt[3]{9}$, $\sqrt[4]{10}$ are surds, but $\sqrt{\pi}$ and $\sqrt{2+\sqrt{17}}$ are not surds because π and $2+\sqrt{17}$ are not rational.

Note

The term "surd" originates from the Latin word for "deaf" or "mute." Historically, Arabian mathematicians classified numbers into two categories: audible (rational) and inaudible (irrational). They referred to irrational numbers as "asamm" in Arabic, which was later translated into Latin as "surdus," giving rise to the term "surd" in English for these irrational numbers.

1.5.2 Operations on Surds

Operations on surds involve manipulating unsimplified radical expressions, such as $\sqrt{2}$ or $\sqrt[3]{5}$, using addition, subtraction, multiplication and division. This helps simplify and solve complex mathematical problems.

1.5.3 Addition and Subtraction of Surds

Similar surds, i.e., surds having same irrational factors, can be added or subtracted into a single term. But remember, $\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$ and $\sqrt{a}-\sqrt{b} \neq \sqrt{a-b}$.

Example 1.12 Simplify by combining similar terms,

$4\sqrt{3}-3\sqrt{27}+2\sqrt{75}$.

Solution:

$$4\sqrt{3}-3\sqrt{27}+2\sqrt{75}$$
$$=4\sqrt{3}-3\sqrt{9\times 3}+2\sqrt{25\times 3}$$
$$=4\sqrt{3}-3\sqrt{9}\sqrt{3}+2\sqrt{25}\sqrt{3}$$
$$=4\sqrt{3}-9\sqrt{3}+10\sqrt{3}$$
$$=(4-9+10)\sqrt{3}=5\sqrt{3}$$

1.5.4 Multiplication and Division of Surds

We can multiply and divide surds of the same order, i.e.,

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}} \text{ and } \sqrt{a}.\sqrt{b}=\sqrt{ab}$$

Example 1.13 Simplify $\dfrac{\sqrt[6]{12}}{\sqrt{3}\times\sqrt[3]{2}}$

Solution:

For $\sqrt{3}\times\sqrt[3]{2}$, the L.C.M of orders 2 and 3 is 6.

Thus, $\sqrt{3}=(3)^{1/2}=(3)^{3/6}=\sqrt[6]{3^3}=\sqrt[6]{27}$ and

$\sqrt[3]{2}=(2)^{1/3}=(2)^{2/6}=\sqrt[6]{(2)^2}=\sqrt[6]{4}$.

Hence,

$$\frac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}=\frac{\sqrt[6]{12}}{\sqrt[6]{27}\sqrt[6]{4}}=\frac{\sqrt[6]{12}}{\sqrt[6]{108}}=\sqrt[6]{\frac{12}{108}}=\sqrt[6]{\frac{1}{9}}$$

Its simplest form is,

$$\sqrt[6]{\left(\frac{1}{3}\right)^2}=\left(\frac{1}{3}\right)^{2/6}=\left(\frac{1}{3}\right)^{1/3}=\sqrt[3]{\frac{1}{3}}$$

Types of Surds

Pure Surd: A surd that can't be simplified further is called a pure surd. For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are pure surds because you can't simplify them any further.

Mixed Surd: A surd that can be broken down into a rational number is called a mixed surd. For instance, $\sqrt{18}$ can be simplified to $3\sqrt{2}$, making it a mixed surd.

On basis of Number of Terms

Monomial surd: A single term under a root, like $\sqrt{2}$ or $3\sqrt{5}$.

Binomial surd: Two under root terms added or subtracted, like $\sqrt{3}+\sqrt{2}$ or $\sqrt{7}-2\sqrt{3}$.

Trinomial surd: Three under root terms added or subtracted, like $\sqrt{2}+\sqrt{3}+\sqrt{5}$ or $2\sqrt{7}-3\sqrt{3}+4\sqrt{2}$.

1.5.5 Rationalization of Surds

Rationalizing surds in math removes roots from the denominator, making it rational by multiplying with an appropriate expression, like the conjugate.

1.5.6 Conjugate of a Surd

The conjugate of a surd is derived by changing the sign in the middle of a binomial surd. So, if you have an expression of the form $a+\sqrt{b}$, where 'a' and 'b' are real numbers and $\sqrt{b}$ is a surd, then the conjugate of this expression would be $a-\sqrt{b}$.

For example, if you have $\sqrt{3}+2$, its conjugate would be $-\sqrt{3}+2$.

1.5.7 Rationalizing a Denominator

Conjugates are particularly useful when rationalizing the denominator of a fraction. When the denominator of a fraction contains a surd, it's often desirable to eliminate it by multiplying the fraction by the conjugate of the denominator. This process is called rationalizing the denominator.

Let us consider an example to understand the concept of rationalization

Example 1.14 Simplify $\frac{2}{3+\sqrt{5}}$

Solution:

To rationalize this, we will multiply the numerator and denominator by the conjugate of the denominator, which is $3-\sqrt{5}$. Remember, to get the conjugate, we simply change the sign between the terms.

$$\frac{2}{3+\sqrt{5}}=\frac{2}{3+\sqrt{5}}\times\frac{3-\sqrt{5}}{3-\sqrt{5}}$$

Multiplying out the numerator, we get

$$2\times\left(3-\sqrt{5}\right)=6-2\sqrt{5}$$

Multiplying out the denominator, we get

$$\left(3+\sqrt{5}\right)\times\left(3-\sqrt{5}\right)=9-5=4$$

So, the rationalized fraction is

$$\frac{6-2\sqrt{5}}{4}=\frac{3}{2}-\frac{\sqrt{5}}{2}$$

Example 1.15

Find rational numbers x and y such that,

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=x+y\sqrt{5}$$

Solution:

We have, $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}$

$$=\frac{4+3\sqrt{5}}{4-3\sqrt{5}}\times\frac{4+3\sqrt{5}}{4+3\sqrt{5}}=\frac{\left(4+3\sqrt{5}\right)^2}{(4)^2-\left(3\sqrt{5}\right)^2}$$

$$=\frac{16+24\sqrt{5}+45}{16-45}=\frac{61+24\sqrt{5}}{-29}$$

$$=\frac{-61}{29}-\frac{24}{29}\sqrt{5}=x+y\sqrt{5}$$

By comparing both sides, we get

$$x=-\frac{61}{29},y=-\frac{24}{29}$$

Example 1.16

Simplify $\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution:

First, we shall rationalize the denominators and then simplify.

We have,

$$\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$$

$$=\frac{6}{2\sqrt{3}-\sqrt{6}}\times\frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}\times\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$=\frac{6\left(2\sqrt{3}+\sqrt{6}\right)}{12-6}+\frac{\sqrt{6}\left(\sqrt{3}-\sqrt{2}\right)}{3-2}-\frac{4\sqrt{3}\left(\sqrt{6}+\sqrt{2}\right)}{6-2}$$

$$=\frac{12\sqrt{3}+6\sqrt{6}}{6}+\frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}}{1}-\frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\cdot\sqrt{2}}{4}$$

$$=2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6}=0$$

Skill 1.3

- Applying the concept of indices and the laws of indices.
- Understanding and applying the concept of radicals and radicands.
- Rationalizing the denominators of radical expressions to eliminate radicals.

Exercise 1.3

1. Express $4\sqrt{11}\left(5\sqrt{2}+2\sqrt{11}\right)$ into simplest from.

2. **Simplify**

 i. $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$ ii. $\left(3\sqrt{2}+2\sqrt{3}\right)^2$

 iii. $\left(\sqrt{12}+\sqrt{3}\right)\left(\sqrt{3}+2\right)$ iv. $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)\left(x+y\right)\left(x^2+y^2\right)$

3. **Find conjugate of the following surds.**

 i. $7-\sqrt{6}$ ii. $9+\sqrt{2}$ iii. $4-\sqrt{15}$

4. **Rationalize the denominator of the following expressions.**

 i. $\dfrac{14}{1-\sqrt{98}}$ ii. $\dfrac{6}{\sqrt{8}+\sqrt{27}}$ iii. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

 iv. $\dfrac{4}{1+\sqrt{5}}$ v. $\dfrac{1}{2\sqrt{3}+\sqrt{5}}$ vi. $\dfrac{3\sqrt{6}}{\sqrt{6}-3}$

5. **Simplify**

 i. $\dfrac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\dfrac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ ii. $\dfrac{2}{\sqrt{3}+1}+\dfrac{4}{\sqrt{3}-2}$

 iii. $\dfrac{2}{\sqrt{5}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{2}}-\dfrac{3}{\sqrt{5}+\sqrt{2}}$ iv. $\dfrac{1}{\sqrt{2}+3}-\dfrac{3}{\sqrt{2}-1}$

6. i. if $x=2-\sqrt{3}$ find the value of $x-\dfrac{1}{x}$ and $\left(x-\dfrac{1}{x}\right)^2$.

 ii. if $x=\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ find the value of $x+\dfrac{1}{x}, x^2+\dfrac{1}{x^2}$ and $x^3+\dfrac{1}{x^3}$.

7. i. Find the value of x^2+4x+4 when $x=2+\sqrt{3}$

 ii. Find the value of $2x^2-3xy$ when $x=\sqrt{2}+3$ and $y=\sqrt{2}-2$

8. Find the rational numbers a and b if,

 i. $\left(8-\sqrt{3}\right)^2=a+b\sqrt{3}$ ii. $\dfrac{10-\sqrt{32}}{\sqrt{2}}=a+b\sqrt{2}$ iii. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}+\dfrac{\sqrt{3}+1}{\sqrt{3}-1}=a+b\sqrt{3}$

9. Write the following in the form $a+b\sqrt{c}$.

i. $\dfrac{1+\sqrt{2}}{3-\sqrt{2}}$ **ii.** $\dfrac{3\sqrt{5}}{3+\sqrt{5}}$ **iii.** $\dfrac{2\sqrt{6}}{\sqrt{6}-2}$

10. The area of a rectangle is $\sqrt{125}$ cm^2. The length of the rectangle is $(2+\sqrt{5})$ cm. Calculate the width of the rectangle and perimeter of the rectangle. Express your answer in the form $a+b\sqrt{5}$, where a and b are integers.

11. Find the length of segment AB in the given figure (see figure 1.1).

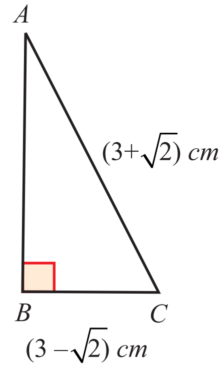
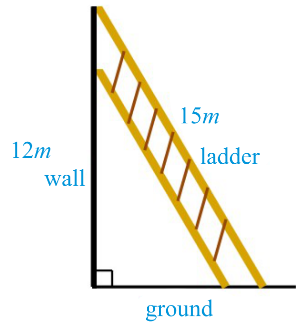


Figure 1.1

12. A square has sides of length x cm and diagonals of length 12 cm. Use Pythagoras' theorem to find the exact value of x and work out the area of the square.

13. An equilateral triangle has sides of length $\sqrt{5}\,cm$. Find the height of the triangle and area of triangle in its simplest surd form.

14. A ladder 13 m long is placed on the ground in such a way that it touches the top of a vertical wall 12 m high. Find the distance of the foot of the ladder from the bottom of the wall.



1.5.8 Exponential and Radical Form

While exponents and radicals may initially seem quite different, they are closely related.

Inverse Operation

Exponents and radicals are inverse operations, reversing each other to return to the original number. For example, $\left(\sqrt{a}\right)^2 = a$.

Interchangeable Notation

Both operations can be expressed interchangeably, a radical in exponent form and vice versa. The nth root of a number x, $\left(\sqrt[n]{x}\right)$ can be written as $x^{\frac{1}{n}}$ and x to the power of $\frac{1}{n}$ $\left(x^{\frac{1}{n}}\right)$ can be written as the nth root of x $\left(\sqrt[n]{x}\right)$.

Properties of Radicals

i. If two or more radicals are multiplied with the same index, you can take the radical once and multiply the numbers inside the radicals, i.e.,

$$\forall\ a,b > 0, \sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{a\times b}$$

ii. If two radicals are in division with the same index, you can take the radical once and divide the numbers inside the radicals, i.e.,

$$\forall\ a \geq 0, b > 0,\ \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 1.17 Simplify $\sqrt[3]{\frac{27a^9}{8b^6}}$

Solution:

$$\sqrt[3]{\frac{27a^9}{8b^6}} = \sqrt[3]{\frac{\left(3a^3\right)^3}{\left(2b^2\right)^3}} = \left(\frac{\left(3a^3\right)^3}{\left(2b^2\right)^3}\right)^{\frac{1}{3}} = \frac{\left(3a^3\right)^{\frac{3}{3}}}{\left(2b^2\right)^{\frac{3}{3}}} = \frac{3a^3}{2b^2}$$

Note

It is just as important to remember that we do not have a sum or difference rule for radicals. That is, in general,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

$$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$$

This is because multiplication and division are closely linked to exponents and radicals, while addition and subtraction lack a direct connection to these operations. Multiplication and division involve repeated multiplication and its reverse, whereas addition and subtraction don't naturally split or combine under radicals in the same way.

iii. Radical of a radical is the same radicand with index multiplied, i.e.,

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$$

For example, $\sqrt[3]{\sqrt[5]{x}} = \sqrt[15]{x}$

iv. Any power of a radical is equal to the radicand whole power divided by index of the radical, i.e.,

$$\left(\sqrt[n]{a}\right)^m = \left(a^{\frac{1}{n}}\right)^m$$

$$\Rightarrow \left(\sqrt[n]{a}\right)^m = \left(a^{\frac{m}{n}}\right)$$

For example, $\left(\sqrt[5]{x}\right)^3 = \left(x^{\frac{1}{5}}\right)^3 = \left(x^{\frac{3}{5}}\right) = \sqrt[5]{x^3}$

v. If power of radicand and radical are same the result is radicand without any power, i.e.,

$$\sqrt[n]{a^n} = a$$

For example, $\sqrt[5]{x^5} = \left(x^5\right)^{\frac{1}{5}} = \left(x^{\frac{5}{5}}\right) = x$

Example 1.18 Simplify $\left(2\sqrt{2}\right)^2$

Solution:

$$\left(2\sqrt{2}\right)^2 = (2)^2\left(\sqrt{2}\right)^2 = 4(2) = 8$$

Example 1.19 Decide which statement is true and which is false.

(a) $\sqrt{4}+\sqrt{12}=\sqrt{16}$ **(b)** $\sqrt{6}\sqrt{3}=\sqrt{18}$

Solution:

(a) is false, because

$\sqrt{4}+\sqrt{12}=\sqrt{4}+\sqrt{4\times 3}=2+\sqrt{4}\times\sqrt{3}=2+2\sqrt{3}=2\left(1+\sqrt{3}\right)\neq 4$ or $\sqrt{16}$

(b) is true, because

$\sqrt{6}\sqrt{3}=\sqrt{6\times 3}=\sqrt{18}$

Example 1.20 Simplify $3\sqrt{425}+4\sqrt{68}$

Solution:

Decompose 425 and 68 into prime factors using synthetic division,

$\sqrt{425}=\sqrt{5\cdot 5\cdot 17}$ $\sqrt{68}=\sqrt{2\cdot 2\cdot 17}$

$\sqrt{425}=5\sqrt{17}$ $\sqrt{68}=2\sqrt{17}$

So, $3\sqrt{425}+4\sqrt{68}=3\left(5\sqrt{17}\right)+4\left(2\sqrt{17}\right)=15\sqrt{17}+8\sqrt{17}=23\sqrt{17}$

Test Yourself

Roots or Powers.

Which of the following expressions are equivalent?

a) $\left(\sqrt[5]{t}\right)^4$ **b)** $\sqrt[4]{t^5}$

c) $\sqrt[5]{t^4}$ **d)** $t^{\frac{4}{5}}$

e) $\left(t^{\frac{1}{5}}\right)^4$

Skill 1.4

- Transforming expressions between radical and exponent forms.
- Simplifying expressions using the laws of exponents.

Exercise 1.4

1. Evaluate each of the following

i. $125^{\frac{4}{3}}$ ii. $512^{\frac{7}{9}}$ iii. $243^{\frac{6}{5}}$ iv. $512^{-\frac{7}{9}}$ v. $343^{-\frac{2}{3}}$ vi. $512^{-\frac{2}{3}}$

2. Write each radical expression in exponential notation and exponential expression in radical notation.

i. $\left(\sqrt[5]{x}\right)^3$ ii. $\sqrt[3]{y^4}$ iii. $\sqrt[3]{-\frac{64}{8}}$ iv. $(-32)^{\frac{2}{5}}$ v. $(8)^{-\frac{4}{3}}$ vi. $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

3. Simplify

i. $\left(x^5\right)^3$ **ii.** $\left(x^3\right)^2 \times x^{3^2}$ **iii.** $x^5 \div x^9$ **iv.** $\sqrt{x^{10}}$ **v.** $\left(x^{-3}\right)^5$

vi. $\sqrt[3]{x^6}$ **vii.** $\left(x^{-1}\right)^2 \times \left(x^{\frac{1}{2}}\right)^8$ **viii.** $3x^2y \times 5x^4y^3$ **ix.** $\sqrt{25x^6y^{-4}}$ **x.** $\sqrt{9x^8y^{-4}} \times \sqrt[3]{8x^6y^{-3}}$

4. Simplify

i. $\sqrt{20}+\sqrt{5}$ **ii.** $\sqrt[3]{24x}-\sqrt[3]{81x}$ **iii.** $\sqrt[4]{4y^3}.\sqrt[4]{12y^2}$ **iv.** $\sqrt[5]{\dfrac{3}{32}}$

v. $\sqrt[3]{-\dfrac{8}{1000}}$ **vi.** $\sqrt[4]{16^3}$

5. Find integers x and y if $2x \times 3y = 6^4$.

6. Find the value of x .

i. $\left(\dfrac{3^3 \times 3^6}{3^7 \times 3^5}\right) = 3^x$ **ii.** $\dfrac{\left(7^x \times 7^3\right)^2}{7^4 \div 7^2} = 7^3$ **iii.** $4^x = \dfrac{1}{64}$ **iv.** $2^x = 0.125$

7. Given that $\dfrac{\left(36x^4\right)^2}{8x^2 \times 3x} = 2^a 3^b x^c$. Find a, b, c.

8. Given that $\dfrac{\sqrt{x^{-1}} \times \sqrt[3]{y^2}}{\sqrt{x^6 y^{-\frac{2}{3}}}} = x^a y^b$. Find a and b.

9. Given that $\dfrac{\sqrt{a^{\frac{4}{5}} b^{-\frac{2}{3}}}}{a^{-\frac{1}{5}} b^{\frac{2}{3}}} = a^x b^y$. Find the value of x and y.

10. Given that $\dfrac{\left(a^x\right)^2}{b^{5-x}} \times \dfrac{b^{y-4}}{a^y} = a^2 b^4$. Find the value of x and y.

11. Simplify $(1+x)^{\frac{3}{2}} - (1+x)^{\frac{1}{2}}$.

12. Simplify

i. $\left(\dfrac{32x^{-6}y^{-4}z}{625x^4yz^{-4}}\right)^{\frac{2}{5}}$ **ii.** $\dfrac{(243)^{-\frac{2}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}}$ **iii.** $\dfrac{2^{\frac{1}{3}} \times (27)^{\frac{1}{3}} \times (60)^{\frac{1}{2}}}{(180)^{\frac{1}{2}} \times (4)^{-\frac{1}{3}} \times (9)^{\frac{1}{4}}}$

iv. $\sqrt{(216)^{\frac{2}{3}} \times \dfrac{(25)^{\frac{1}{2}}}{(0.04)^{-\frac{1}{2}}}}$ **v.** $\sqrt[3]{\dfrac{a^l}{a^m}} \times \sqrt[3]{\dfrac{a^m}{a^n}} \times \sqrt[3]{\dfrac{a^n}{a^l}}$ **vi.** $\dfrac{\left(\sqrt{98a^6b^5c^4} + \sqrt{63a^3b^2c} - \sqrt{27a^5bc^3}\right)}{\sqrt{126a^4b^3c^2} + \sqrt{7abc^4}}$

Knowledge 1.6 Applications of Rational Numbers in Financial Arithmetic

Rational numbers are vital for everyday problem-solving in various fields, including inventory management, temperature conversions, banking, finance and percentage calculations, contributing to effective decision-making and financial management.

Student Learning Outcomes

- Apply concepts of rational numbers to real word problems (such as inventory (stock taking), temperature, banking, measures of gain and loss, sources of income and expenditure).

Inventory Management

Inventory management is like smartly organizing and keeping track of stuff a business has. It helps make sure there's enough, but not too much, so things run smoothly and costs stay low

Example 1.21 A bookstore received a shipment of 1,000 books. After selling 65% of them, they found that 150 books were damaged and had to be discarded. What percentage of the books is left in the store?

Solution:

Let's break this problem down step by step:

Books Sold:

The bookstore sold 65% of the 1,000 books.

So, the number of books sold

$$= \frac{65}{100} \times 1{,}000 \ = 650 \text{ books}$$

Books Discarded:

The number of discarded books is 150.

Books Left:

The number of books left in the store = Total books − Books sold − Books discarded

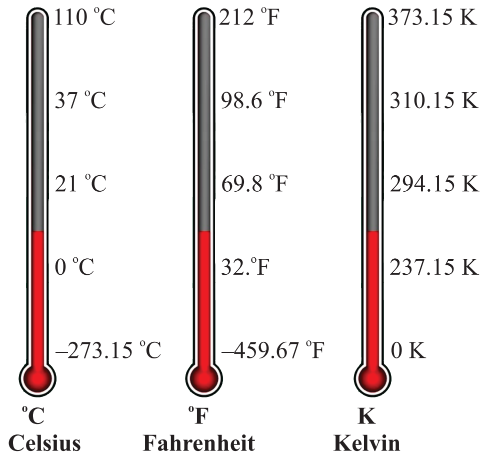
So, books left = 1,000 (Total) − 650 (Sold) − 150 (Discarded) = 200 books

Percentage of Books Left:

$$\text{The percentage of books left in the store} = \frac{\text{Number of books left}}{\text{Total number of books}} \times 100$$

$$\text{So, the percentage left} = \frac{200}{1000} \times 100 = 20\%$$

Hence, 20% of the books remain after selling 65% and discarding 150 damaged books.



Temperature Conversions

Temperature conversions between Celsius, Fahrenheit and Kelvin use specific formulas.

Celsius to Fahrenheit:

$$\text{Fahrenheit}(°F) = \left(\text{Celsius}(°C) \times \frac{9}{5}\right) + 32$$

Fahrenheit to Celsius:

$$\text{Celsius}(°C) = \left(\text{Fahrenheit}(°F) - 32\right) \times \frac{5}{9}$$

Celsius to Kelvin:

$$\text{Kelvin}(K) = \text{Celsius}(°C) + 273.15$$

Kelvin to Celsius:

$$\text{Celsius}(°C) = \text{Kelvin}(K) - 273.15$$

Fahrenheit to Kelvin:

$$\text{Kelvin}(K) = \left(\text{Fahrenheit}(°F) + 459.67\right) \times \frac{5}{9}$$

Kelvin to Fahrenheit:

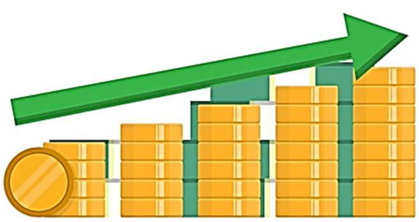
$$\text{Fahrenheit}(°F) = \text{Kelvin}(K) \times \frac{9}{5} - 459.67$$

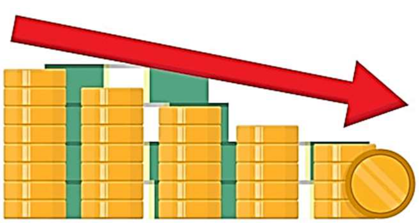
Example 1.22 Convert 100 degrees Celsius to Fahrenheit and Kelvin.

Solution:

$$\text{Fahrenheit }(°F) = \left(100 \times \frac{9}{5}\right) + 32 = 212°F$$

$$\text{Kelvin }(K) = 100 + 273.15 = 373.15\text{ K}$$





Profit and Loss

Profit and loss are vital in finance and business. Gain is the positive difference (profit) between selling and cost prices, while loss is the negative difference. Both can be expressed as percentages with these formulas,

Profit:
Profit = Selling Price – Cost Price
Loss:
Loss = Cost Price – Selling Price
Profit Percentage:

$$\text{Profit Percentage} = \left(\frac{\text{Profit}}{\text{Cost Price}}\right) \times 100\%$$

Loss Percentage:

$$\text{Loss Percentage} = \left(\frac{\text{Loss}}{\text{Cost Price}}\right) \times 100\%$$

Example 1.23 A retailer, Zahid, owns a clothing store in Pakistan. He bought various clothing items from different suppliers and sold them to customers. Calculate his overall profit or loss based on the following transactions:

1. Zahid bought 10 shirts at 1,500 PKR each.
2. He bought 20 trousers at 2,000 PKR each.
3. Zahid sold 7 shirts at 2,000 PKR each.
4. He sold 15 trousers at 3,000 PKR each.

Zahid also had to pay 500 PKR for transportation costs related to the purchases.
Calculate Zahid's total profit or loss in rupees and as a percentage of his total investment.

Solution:

- Cost Price of Shirts: 10 shirts × 1,500 PKR = 15,000 PKR
- Cost Price of Trousers: 20 trousers × 2,000 PKR = 40,000 PKR
- Selling Price of Shirts: 7 shirts × 2,000 PKR = 14,000 PKR
- Selling Price of Trousers: 15 trousers × 3,000 PKR = 45,000 PKR
- Transportation Charges 500 PKR

Total Cost Price:

15,000 PKR (shirts) + 40,000 PKR (trousers) + 500 PKR (transportation) = 55,500 PKR

Total Selling Price:

14,000 PKR (shirts) + 45,000 PKR (trousers) = 59,000 PKR

Now, we can calculate the profit or loss:

Profit/Loss = Total Selling Price − Total Cost Price

Profit/Loss = 59,000 PKR − 55,500 PKR = 3,500 PKR

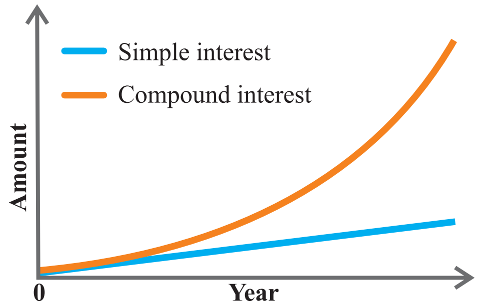
Since the selling price is higher than the cost price, Zahid made a profit of 3,500 PKR.

$$\text{Profit Percentage} = = \left(\frac{\text{Profit}}{\text{Cost Price}}\right) \times 100\%$$

$$\text{Profit Percentage} = \left(\frac{3,500}{55,500}\right) \times 100 \approx 6.31\%$$

Compound Interest

Compound interest calculates interest on the initial principal and any accumulated interest, leading to increased total interest over time. Compounding at regular intervals (e.g., annually, monthly) significantly affects the total amount of interest.



Compound interest causes values to grow exponentially, multiplying over time, whereas simple interest leads to linear growth, with values increasing through straightforward addition.

The formula for compound interest is: $A = P\left(1+\frac{r}{n}\right)^{nt}$,

where:

A: Final amount with interest P: Initial principal amount

r: Annual interest rate (decimal)

n: Compounding frequency per year

t: Investment/borrowing time in years

Example 1.24 A sum of PKR 500,000 is deposited in a bank account which offers an annual interest rate of 4% compounded annually. How much will the amount be after 3 years?

Solution:

$r = 4\% = 0.04$, $n = 1$ (as interest is compounded annually) $t = 3$ years

Substituting these into the formula, we get:

$$A = 500{,}000\left(1+\frac{0.04}{1}\right)^{1\times 3}$$

$$A = 500{,}000(1.04)^3 \Rightarrow A = \text{PKR } 562{,}432.64$$

So, after 3 years the amount will be PKR 562,432.64.

Simulation

Visualize how compound interest varies exponentially.
Working: Drag the red point and see the variation.

Skill 1.5

Applying mathematical concepts to solve real-world problems, specifically using rational numbers in contexts such as banking, gains and losses etc.

Exercise 1.5

1. A bookstore in Lahore started with 5000 Urdu novels. In the first month, it sold $\frac{3}{5}$ of its stock. During the next month, it sold $\frac{1}{4}$ of the remaining stock. How many novels are left in the bookstore?

2. A supermarket in Islamabad had an initial inventory of 1500 kg of Basmati rice. During Ramzan, it sold $\frac{9}{10}$ of its stock. After Ramzan, they restocked $\frac{2}{3}$ of the sold quantity. How many kg of rice do they have now?

3. A Pakistani textile factory produces 1200 meters of fabric every day. Each day they use $\frac{1}{5}$ of the fabric produced on that day plus 10% of the remaining fabric from the previous day's production. How much fabric is left after 7 days?

4. A toy store initially has 2,500 toys in stock. Each day, they sell 7% of the remaining toys and add 150 new toys to their stock. How many toys are left after one week and how many toys they had sold in one week?

5. A scientist is working with a temperature of 310 K. Convert this temperature to Celsius and Fahrenheit.

6. The mean temperature for January in Muree is -1 degree Celsius, while in July it's 18 degrees Celsius. In Kashmir, the mean temperature for January is 56 degrees Fahrenheit, while in July it's 104 degrees Fahrenheit. Calculate these temperatures in the opposite scale and determine which city has a larger temperature range.

7. Samina has a checking account balance of PKR 20,000. During the month, she wrote checks for PKR 15,750 and made a deposit of PKR 7,500. She also used her debit card for purchases totaling PKR 1,250. What is her account balance at the end of the month?

8. Jamil borrowed PKR8,000 from his friend and promised to repay PKR 9,500 after a year. Calculate the simple interest rate.

9. Fatima takes a loan of PKR 500,000 from a bank that charges 7% annual interest, compounded annually. If she wants to repay the loan in 5 years, how much will she have to pay in total?

10. Ahmed invests PKR 10,000 in a bond that pays 8% simple interest annually. How much will he earn in interest over a 4-year period?

11. A Businessman took a loan of PKR 30,00000 from the bank at 12% annual interest. If He plans to repay the loan in 5 equal annual installments, how much will each installment be?

12. A property was bought for PKR 750,0000, and after some improvements costing PKR 950000, it was sold for PKR 1,200,0000. Calculate the gain percentage.

13. A retailer bought 100 units of a product at PKR 50 each. The retailer sold 70 units of the product at a price of PKR 80 each and the remaining 30 units at a clearance price of PKR 30 each. Calculate the overall profit or loss percentage for the retailer.

14. Fatima earns PKR 510,000 per month. She spends 153,000 PKR (30% of her income) on rent, 127,500 PKR (25% of her income) on food, 51,000 PKR (10% of her income) on transportation and 102,000 PKR (20% of her income) on other bills. How much money does Fatima save each month Also write your answer in Percentage of the total income.

Review Exercise 1

1. Identify True or False

i. For any real numbers a and b the trichotomy property asserts that exactly two of the following is true: $a > b$ or $a = b$ or $a < b$.

ii. The commutative property of multiplication and addition means that the order in which numbers are added or multiplied does not affect the result.

iii. Rational numbers can always be expressed as a non-terminating, non-repeating decimal expansion.

iv. The power rule for exponents can be expressed as $(a^m)^n = a^{mn}$.

v. The closure property of real numbers states that the subtraction of any two real numbers is always a real number.

vi. Simplifying expressions involving nth roots and radical expressions with variables involves adding the exponents together.

vii. The additive property of inequalities allows you to add the same quantity to both sides of an inequality without changing its direction.

viii. Rationalizing denominators of radical expressions involves making the denominator a rational number by removing any radicals.

ix. The symmetric property of real numbers asserts that if a is greater than b, then b must also be greater than$a$.

x. The inverse elements for addition and multiplication in real numbers are the opposite and the reciprocal, respectively.

2. Four every question, there are four options, choose the right one.

i. Real numbers consist of:

a. Only rational numbers
b. Only irrational numbers
c. Both rational and irrational numbers
d. Neither rational nor irrational numbers

ii. Both addition and multiplication in $\mathbb{Z}$ are __________.

a. Commutative
b. Associative
c. Distributive
d. All of these

iii. The decimal expansion of $\dfrac{63}{72\times 175}$ is:

a. terminating **b.** non-terminating **c.** non-terminating and repeating **d.** None of these

iv. A rational number between $\dfrac{1}{2}$ and $\dfrac{2}{7}$ is

a. $\dfrac{1}{14}$ **b.** $\dfrac{2}{21}$ **c.** $\dfrac{5}{14}$ **d.** $\dfrac{5}{21}$

v. The number 1.101001000100001... is

a. a natural number **b.** a whole number

c. a rational number **d.** an irrational number

vi. Out of the four numbers

(i) $\left(\sqrt{5}-\dfrac{1}{\sqrt{5}}\right)^3$ **(ii)** $2.123\overline{123}$ **(iii)** $2.100100\ldots$ **(iv)** $\left(2\sqrt{3}-\sqrt{2}\right)\left(2\sqrt{3}+\sqrt{2}\right)$, the irrational number is

a. (i) **b.** (ii) **c.** (iii) **d.** (iv)

vii. If $a=b$ and $a\neq 0,\ b\neq 0$, which of the following is true?

a. $\dfrac{1}{a}=\dfrac{1}{b}$ by the division property of equality. **b.** $\dfrac{1}{a}\neq\dfrac{1}{b}$ by the subtraction property of equality.

c. $a=\dfrac{1}{a}$ by the reflexive property of equality. **d.** None of the above.

viii. $x = 3^2\times 2^3$ then $x^4=$

a. $3^2\times 2^3$ **b.** $3^6\times 2^7$ **c.** $3^8\times 2^3$ **d.** $3^8\times 12^{12}$

ix. Suppose $x+5=y+5$ and $y=z$. What property would allow you to say that $x+5=z+5$?

a. Substitution Property **b.** Reflexive Property

c. Symmetric Property **d.** Transitive Property

x. $\dfrac{3}{9-\sqrt{5}}=$

a. $\dfrac{27+3\sqrt{5}}{4}$ **b.** $\dfrac{27-3\sqrt{5}}{76}$ **c.** $\dfrac{27+3\sqrt{3}}{76}$ **d.** $\dfrac{1}{3}-\dfrac{3}{\sqrt{5}}$

xi. A store sells a product for PKR 50 per unit. The cost of the product is PKR 30 per unit. If the store has 100 units in inventory and sells 60 units, what is the store's gross profit?

a. PKR 1200 **b.** PKR 2000 **c.** PKR 3000 **d.** PKR 4000

xii. The cost price of 20 articles is the same as the selling price of x articles. If the profit is 25%, then the value of x is

a. 25 **b.** 18 **c.** 15 **d.** 16

xiii. According to the Multiplicative Identity, $(x + 7) \times$____ $=$____. Which choice shows the correct blank entries (in order)?

a. 0, $(x + 7)$ **b.** 0, 0 **c.** 1, $(x + 7)$ **d.** 1, 1

xiv. Given that $a \neq 0$ and $b \neq 0$, and $\frac{a+b}{a-b} = 1$, what can we infer about a and b?

a. $a = b$ **b.** $a = -b$ **c.** $a > b$ **d.** $a < b$

xv. If p, q, r are real numbers and $p > q > r > 0$, which is the smallest?

a. $\frac{1}{p}$ **b.** $\frac{1}{q}$ **c.** $\frac{1}{r}$ **d.** None of the above

3. Find the value of x in each of the follow

i. $2^{2x+2} = 128$ **ii.** $\frac{1}{4^{-x}} = \frac{1}{2}$ **iii.** $3^{-x+4} = 81$ **iv.** $8^{-3x} = \frac{1}{4}$

4. Solve the following expression for y $\frac{36^{2y-5}}{6^{3y}} = \frac{6^{2y-1}}{216^{y+6}}$

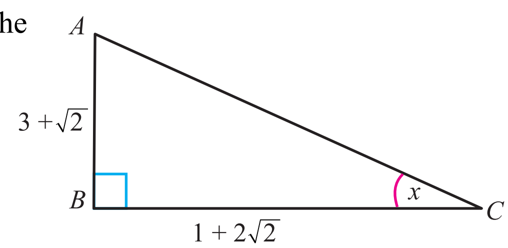
5. Simplify $(1+x)^{\frac{3}{2}} \times (1+x)^{\frac{1}{2}}$

6. Simplify each of the following

i. $\sqrt[3]{x} \times \frac{\sqrt{x^6}}{x^{-\frac{1}{3}}}$ **ii.** $\frac{\sqrt[3]{x} \times \sqrt[3]{x^5}}{x^{-2}}$ **iii.** $\frac{x^2 \times \sqrt{x^5}}{x^{-\frac{1}{2}}}$ **iv.** $\frac{(3xy)^2 \times \sqrt{x^4 y^6}}{(2x^4 y^3)^2}$

7. Express $6\left(1+\sqrt{3}\right)^{-2}$ in the form $a+b\sqrt{3}$, where a and b are integers to be found.

8. Find the area and perimeter of the triangle. Find your answer inthe form of $a+b\sqrt{2}$.



9. Simplify $\frac{1}{\sqrt{a^2 - x^2} + a} + \frac{1}{a - \sqrt{a^2 - x^2}}$

10. If $x = \sqrt{2} - 3$ find

i. $x + \frac{1}{x}$ **ii.** $x - \frac{1}{x}$ **iii.** $x^2 + \frac{1}{x^2}$ **iv.** $x^2 - \frac{1}{x^2}$

Student Review Check List

1. ▢ Historical Development of Number Systems

a. Understand the evolution through civilizations.

2. ▢ Classification of Numbers

a. Real Numbers (Includes all rational and irrational numbers)

b. Rational Numbers (can be expressed as $\frac{a}{b}$ and $a,b \in \mathbb{Z}$ where $b \neq 0$)

c. Irrational Numbers (cannot be expressed as a simple fraction)

3. ▢ Properties of Real Numbers:

a. Closure ($a+b$ and ab are real for $a,b \in \mathbb{R}$)

b. Identity Element ($a+0=a, a\times 1=a$)

c. Inverse Element ($a+(-a)$=0, $a\times\frac{1}{a}=1$, for $a\neq0$)

d. Commutativity $(a+b=b+a\ ,ab=ba)$

e. Associativity

f. $(a+b)+c=a+(b+c),(ab)c=a(bc)$

Distributivity $a(b+c)=ab+ac$

4. ▢ Properties of Equality for Real Numbers

a. Reflexive ($a=a$)

b. Symmetric (If $a=b$, then $b=a$)

c. Transitive (If $a=b$ and $b=c$ then $a=c$)

d. Substitution (If $a=b$, then a can replace b in any expression)

5. ▢ Properties of Inequality for Real Numbers

a. Transitive (If $a>b$ and $b>c$, then $a>c$)

b. Trichotomy ($a<b$, $a=b$, or $a>b$)

c. Additive (If $a>b$, then $a+c>b+c$)

d. Multiplicative (If $a>b$ and .$c>0$., then $ac>bc$ and if $c<0$ then $ac<bc$)

e. Cancellation (if $a+c<b+c$ then $a<c$, if $ac<bc$, then $a<b$)

6. ▢ Concepts Related to Exponents and Radicals

a. Indices (Exponents: a^n)

b. Radicals ($\sqrt[n]{a}$)

c. Radicands (The value inside the radical symbol)

d. Surds (Roots that result in irrational numbers)

e. Conjugate (Changing the sign of radical)

f. Rationalizing a Surd (multiplying the fraction with conjugate of denominator)

7. ▢ Laws of Indices and Relationships

a. Product Rule ($a^m \times a^n = a^{m+n}$)

b. Quotient Rule ($\frac{a^m}{a^n}=a^{m-n}$)

c. Power Rule ($(a^m)^n=a^{mn}$)

8. ▢ Real-World Applications of Rational Numbers

a. Understand applications in various contexts like banking, temperature conversions, etc.

CHAPTER

2

Logarithm

Logarithms: The Secret Behind Predicting Growth in Science

Did you know that logarithms are super useful for figuring out how quickly things grow or shrink over time, like in science experiments with bacteria? Imagine you've got a tiny bunch of bacteria that doubles in size every hour. If you start with just a little bit, they can fill up a whole petri dish in no time! By using logarithms, scientists can work out how fast the bacteria will grow without having to watch them double and double again for hours on end. This is because logarithms help turn that doubling (which happens again and again) into a simple problem, kind of like solving a puzzle. So, if you're curious about how long it will take for those tiny bacteria to take over the dish, logarithms are your best friend. This makes it super easy for scientists to predict stuff, like how diseases spread or how fast a plant will grow, just by using this cool math trick!

Students' Learning Outcome

1. Express a number in scientific notations and vice versa.
2. Describe logarithm of a number.
3. Differentiate between common and natural logarithm.
4. Apply laws of logarithm to real life situations such as growth and decay, loudness of sound.

Pre & Post Requisite

Class 8
Chapter # 2
Estimation and Approximation

Class 9 (physics)
Chapter # 1
Scientific Notation

Class 9
Chapter # 1
Logarithm

Knowledge

① **Understanding Scientific Notation:**
- Comprehend the concept of expressing large numbers in scientific notation for simplification and clarity.
- Recognize how to convert numbers from scientific notation back to standard form.

② **Fundamentals of Logarithms:**
- Grasp the definition and the mathematical significance of the logarithm of a number.
- Identify the characteristics and differences between common logarithms (base 10) and natural logarithms (base e).

③ **Distinction Between Logarithm Types:**
- Distinguish between common logarithms and natural logarithms, noting their specific bases and typical applications.

④ **Applications of Logarithms:**
- Understand how logarithms are used in real-life situations, such as modeling exponential growth and decay in populations, finances or radioactive substances.
- Learn about the application of logarithms in measuring the loudness of sound in decibels and their role in acoustic engineering.

Skills

- Employ scientific notation in various contexts to simplify numerical expressions and calculations.
- Transition seamlessly between scientific notation and standard number form.
- Converting logarithms into exponential and vice versa.
- Applying definition of log to find the unknowns.
- Choose appropriately between common and natural logarithms based on the context of the problem.
- Utilizing the laws of logarithms to combine or separate logarithmic terms.
- Utilizing the laws of logarithms to evaluate logarithmic equations.
- Use logarithmic functions to analyze real-life situations such as population growth, radioactive decay, and sound intensity levels.

History

In 1614, Scottish mathematician John Napier introduced logarithms, a breakthrough simplifying multiplication and division, speeding up calculations before calculators existed. This invention transformed mathematics, aiding scientific discoveries and advancing navigation, marking a significant impact on the development of modern mathematics.

Introduction

Logarithms have been instrumental in the evolution of mathematics and its myriad applications, notably through the enhancements by Henry Briggs. These refinements significantly broadened the utility of logarithms, making complex calculations more manageable across various fields such as science, engineering, and navigation. By simplifying the processes of multiplication and division into more straightforward operations, logarithms enabled significant advancements in technology and scientific understanding. The work of Briggs, in particular, made logarithmic methods more accessible, cementing their role as a cornerstone in the development of modern computational techniques and our exploration of the natural world.

Student Learning Outcomes

- Express a number in scientific notations and vice versa.

Knowledge 2.1 Scientific Notation

Scientific notation is developed to efficiently represent large and small numbers. It simplifies handling large or small values with many zeros and facilitates calculations in scientific and technical fields.

Scientific notation is a concise mathematical representation that expresses numbers as a product of a significant digit and a power of 10 exponent.

For example, 45×10^{-3}, $0.0367\times10^{+4}$, 7.3×10^{-3}, $3.21\times10^{+4}$

2.1.1 Standard Form

Any number X can be written as power of 10 as $X = A\times10^n$ where n is the power of ten and A is non-zero digit ranging $1\le A<10$.

Positive Power of Ten

To create a positive power of 10, move the decimal point to the left; the exponent is determined by the count of places the decimal has shifted. In the example below, shifting two digits left gives 10^{+2}.

$$\boxed{475.31=4.7531\times10^{+2}}$$

Similarly, $70100000 = 7.01\times10^7$, $34860 = 3.486\times10^4$, $54000 = 5.4\times10^4$

Negative Power of Ten

To form a negative power of 10, shift the decimal point to the right. The value of the exponent is equal to the total number of positions the decimal has been moved. In the example below, shifting the point three digits left gives 10^{-3}.

$$0.00325 = 3.25\times10^{-3}$$

Similarly, $0.0308 = 3.08\times10^{-2}$, $0.00097 = 9.7\times10^{-4}$, $0.0006741 = 6.741\times10^{-4}$

2.1.2 Ordinary Form

By ordinary form we mean to eliminate power of ten from scientific notation.

Eliminating Positive Powers

To create a positive power of ten, we move the decimal point to the left; to counteract a positive power, we shift the decimal point to the right. Let's consider an example where shifting the decimal point two digits to the right eliminates a +2 power.

$$4.7531\times10^{+2} = 475.31$$

Eliminating Negative Power

To create a negative power of ten, we shift the decimal point to the right; to eliminate a negative power, we move the decimal point to the left. Consider the following example, where shifting the decimal point three digits to the left eliminates a −3 power.

$$3.25\times10^{-3} = 0.00325$$

2.1.3 Arithmetic Operations on Scientific Notations

1. Multiplication Rule:

Expression: $(a \times 10^m) \times (b \times 10^n) = (a \times b) \times 10^{n+m}$

Explanation: To multiply numbers in scientific notation, multiply their significant figures and add their exponents of 10 to get the product's scientific notation form.

2. Division Rule:

Expression: $\dfrac{(a\times10^m)}{(b\times10^n)} = \left(\dfrac{a}{b}\right)\times10^{m-n}$

Explanation: When dividing numbers in scientific notation, divide their significant figures and subtract the exponent of the divisor from the exponent of the dividend to obtain the quotient's scientific notation form.

Class Activity

This headline appeared in a local newspaper: "A significant portion of Pakistanis eat at biryani restaurants every day." Analyze the following information to estimate the percentage of Pakistanis engaging in this dining habit:

- There are about 1×10^5 biryani Restaurants in Pakistan.
- Each restaurant serves about 2.5×10^2 people every day.
- The population of Pakistan is about 2.4149×10^8

Example 2.1

Evaluate the result of multiplying $(6.8\times10^{5})\times(2.5\times10^{-3})$.

Solution:

Step 1: Multiply the significant figures: $6.8\times 2.5=17$

Step 2: Add the exponents of 10: $5+(-3)=2$

The product 17×10^{2} is in scientific notation.

Example 2.2

Perform the division of $\dfrac{(7.2\times10^{4})}{3\times10^{2}}$.

Solution:

Step 1: Divide the significant figures: $\dfrac{7.2}{3}=2.4$

Step 2: Subtract the exponents of 10: $4-2=2$.

The quotient 2.4×10^{2} is in scientific notation.

3. Addition and Subtraction Rule:

Addition $(a\times10^{n})+(b\times10^{n})=(a+b)\times10^{n}$

Subtraction $(a\times10^{n})-(b\times10^{n})=(a-b)\times10^{n}$

Explanation: For addition or subtraction, align the exponents of 10 for all numbers, adjust the significands accordingly and perform the operation while maintaining the common exponent. Alternatively, convert them to ordinary form for regular addition or subtraction and later switch back to scientific notation.

Example 2.3

Perform the subtraction of $(5.3\times10^{-3})-(4.9\times10^{-4})$.

Solution:

First adjust the power of 1st value to -4 or adjust the power of 2nd value to -3.

Step 1: We are adjusting the 1st value, $5.3\times10^{-3}=53\times10^{-4}$

Step 2: Take 10^{-4} common, $10^{-4}(53-4.9)$

Step 3: Perform the subtraction: $53-4.9=48.1$

The resultant 48.1×10^{-4} is in scientific notation.

Skill 2.1

- Employ scientific notation in various contexts to simplify numerical expressions and calculations.
- Transition seamlessly between scientific notation and standard number form.

Exercise 2.1

1. Write the following numbers in standard form.

   i. 723000 ii. 53400 iii. 6934390 iv. 412300000

   v. 37.42 million vi. 0.000082 vii. 0.0060 viii. 0.00000000056

2. Write the following in ordinary form.

   i. 6.9×10^{6} ii. 6.07×10^{-4} iii. 4.73×10^{4} iv. 2.79×10^{7}

   v. 4.83×10^{-5} vi. 2.61×10^{-6} vii. 3.69×10^{3} viii. 6.07×10^{-4}

3. Write the following in standard form.

   i. 68×10^{-5} ii. 720×10^{6} iii. 8×10^{5} iv. 0.75×10^{7}

   v. 0.4×10^{-10} vi. 50×10^{-6}

4. Write these numbers in order of magnitude starting with the largest.

   $3.2\times 10^{-4}, 6.8\times 10^{-5}, 5.57\times 10^{-9}, 5.8\times 10^{-7}, 6.741\times 10^{-4}, 8.414\times 10^{2}$

5. Deduce the value of n in each of the following cases.

   i. $0.00025 = 2.5\times 10^{n}$ ii. $0.00357 = 3.57\times 10^{n}$ iii. $0.0000006 = 6\times 10^{n}$

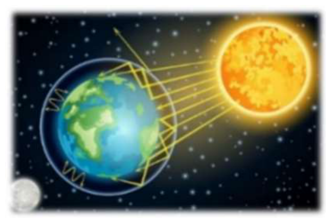
   iv. $0.004^{2} = 1.6\times 10^{n}$ v. $0.00065^{2} = 4.225\times 10^{n}$ vi. $0.0002^{n} = 8\times 10^{-12}$

6. Find the value of the following and write your answer in scientific notation.

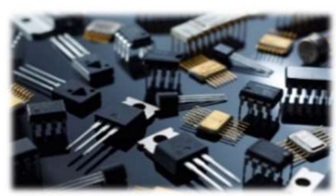
   i. $(9.6\times 10^{11}) \div (2.4\times 10^{5})$ ii. $3.15\times 10^{-9} \div 7.0\times 10^{6}$ iii. $8.5\times 10^{-6}\times 6\times 10^{15}$

   iv. $3.8\times 10^{5}\times 4.6\times 10^{4}$ v. $6.6\times 10^{7} - 4.9\times 10^{6}$ vi. $4.07\times 10^{7} - 5.1\times 10^{6}$

7. Population of Pakistan according to latest digital census is 241.49 million, convert this in standard form.



8. Light from the sun takes approximately 8 minutes to reach Earth. If light travels at a speed of 3×10^{8} m/s. Calculate to 3 significant figures the distance from sun to Earth using $S = vt$.



9. A computer chip has 2.1×10^{9} transistors. If a new version of the chip has 4.5×10^{9} transistors, how many more transistors does the new version have?

10. A certain species of bacteria doubles in population every 2 hours. If there are initially 1.5×10^{3} bacteria, how many bacteria will there be after 100 hours?

Knowledge 2.2 Logarithm

Logarithm, in mathematics, is another word for the exponentiation. It determines the power to which a number must be raised to obtain another number.

Student Learning Outcomes

- Describe logarithm of a number.

The logarithm of a positive real number a with respect to a base b (which is also a positive real number different from 1), is defined as the exponent x to which b must be raised to obtain a.

$$b^x = a \Leftrightarrow log_b\, a = x, \text{ where } b > 0, b \neq 1 \text{ and } a > 0.$$

Equals to

$$log_b(a) = x$$

Raised to

This relationship indicates that logarithms are the inverse operations of exponentiation, Here "a" is called the argument which is inside the log and "b" is called the base which is at the bottom of the log.

The logarithm answers the question: How many times must one number be multiplied to obtain another?

For example, how many 2's are multiplied to get the answer 128?

If we multiply 2 for 7 times, we get the answer 128.

Therefore, the logarithm of 128 with respect to base 2 is 7.

The logarithm form can be written as

$$log_2 128 = 7 \cdots\cdots\cdots (i)$$

The above logarithm form can also be written as:

$$2^7 = 128 \cdots\cdots\cdots (ii)$$

Here, the equations (i) and (ii) both represent the same meaning.

Examples of conversion	
Exponents	Logarithms
$6^2 = 36$	$\log_6 36 = 2$
$10^2 = 100$	$\log_{10} 100 = 2$
$3^3 = 27$	$\log_3 27 = 3$

2.2.1 Understanding the same concept with number line representation

Take a geometric sequence with common ratio "3" on number line 1 line and a general arithmetic sequence with common difference as "1" on number line 2. The exponents in the geometric sequence terms with respect to base 3 are basically the logs of the geometric sequence which are plotted on 2nd number line (see figure 2.1).

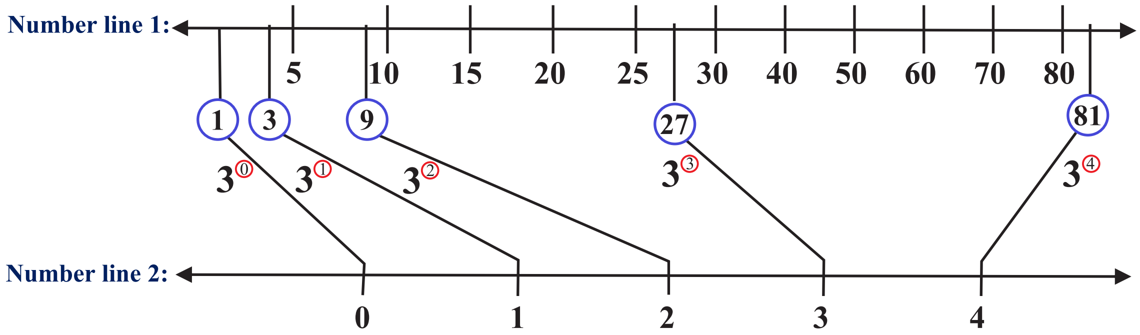


Figure 2.1

Number line 1: Displays a number sequence within a blue circle, where each term increases by a factor of three

Number line 2: Displays the exponents for the numbers highlighted in the first line, where the logarithm of a number encircled in blue is the exponent(encircled red) to which 3 must be raised to yield that numbered circle)

2.2.2 Important Logarithmic Understandings

1. Logarithm of a Positive Number can be Negative

A negative answer in log means that we will use a negative exponent (power) when we convert from log form to exponential form. In exponential form, a negative exponent implies a division.

For instance, a^{-n} is equivalent to $\frac{1}{a^n}$.

Negative logarithms typically arise when the base is positive and the input falls between 0 and 1.

Test Yourself

Identify as true or false

1. $log(1000)$ in any base is always 3.

In the expression

2. $log_2 4$ and $log_4 2$ are reciprocals of each other.

Example 2.4

Find x if $log_5 \frac{1}{25} = x$.

Solution:

We can convert from log form to exponential form to get

$$5^x = \frac{1}{25} \Rightarrow 5^x = 5^{-2}.$$

$$x = -2$$

Equal bases imply equal exponents, resulting in a negative logarithm in this case.

2. Logarithm of a Negative Number

You cannot take the log of a negative number (unless you want to deal with complex numbers). Once again, converting from log form to exponential form will help us to see why this is the case.

For example, let's say you wanted to take the log of -100, using a base of 10. So, we want to solve this log form equation for x:

$$log_{10}(-100) = x$$

Converting from log form to exponential form, we get

$$10^x = -100$$

The base is 10 (which is positive), while the argument is -100 (which is negative). This means that there is no real number we can substitute for x to get a result of -100 (any real number we put in for x will give us a positive value).

Note

- Log of zero and a negative number is undefined.
- Log of 1 is always zero with respect to any base.
- Base of logarithm can be any non-negative real number except zero and 1.
- Argument of log is always a positive real number.
- Argument of the log can never be negative and zero because the result is undefined.
- For argument between 0 and 1 logarithm is always negative.

3. Concept of Negative Base

A logarithm cannot have a negative base (unless we are dealing with complex numbers).

4. Logarithm of "1"

A logarithm can be zero if the argument to the log function is 1 and this is true for any valid base. We can see this easily by converting the log form to exponential form.

$$log_b\, a = 0$$

$$b^0 = a$$

Since $b^0 = 1$, so

$$a = 1$$

5. Undefined Logarithm

Note

Complex numbers are a type of number that include the square root of negative one, represented as i, allowing us to solve equations with negative square roots and work with numbers beyond the usual real number system

The log function is undefined for zero or negative arguments. While negative arguments can be used with complex numbers, an argument of zero is always illogical in a log function. Once again, we'll convert from log form to exponential form to see why,

$$log_2\, 0 = x$$

Converting to exponential form, we get,

$$2^x = 0$$

There is no number we can substitute for x to make this equation true. Any real value of x we choose will give us a positive number.

Let us the consider the following examples to comprehend the concept of logarithms.

Example 2.5

Convert from exponential to logarithmic and vice versa.

i. $2^4 = x$ **ii.** $log_7\, 49 = 2$ **iii.** $log_4 \frac{1}{16} = y$

Solution:

i. $log_2\, x = 4$ **ii.** $7^2 = 49$ **iii.** $4^y = \frac{1}{16}$

Example 2.6

Find the unknown variable.

i. $2^{3x} = 8$ ii. $\log x^2 = 4$ iii. $\log_9 \frac{1}{81} = y$

Solution:

i. $2^{3x} = 8$

$2^{3x} = 2^3$

Comparing,

$3x = 3$

$x = 1$

ii. $\log x^2 = 4$

Common log has a base 10 so,

$10^4 = x^2$

$\sqrt{10^4} = \sqrt{x^2}$

$10^2 = x$

$x = 100$

iii. $\log_9 \frac{1}{81} = y$

$9^y = \frac{1}{81}$

$9^y = 9^{-2}$

$9^y = \frac{1}{9^2}$

$y = -2$

How can one manually determine the integers between which a logarithm lies?

Example 2.7

Write the two consecutive integers between which $\log_4 100$ lies.

Solution:

Since, $4^3 = 64$ and $4^4 = 128$. 100 lies somewhere between 64 and 128. So, by understanding of exponentiation nature of logarithm we can say that $\log_4 100$ will lie somewhere between 3 and 4 (greater than 3 and less than 4).

Knowledge 2.3 Natural Logarithm

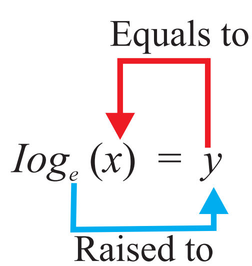
Student Learning Outcomes

- Differentiate between common and natural logarithm of a number.

A natural logarithm is a logarithm that uses the mathematical constant *"e"* (which is approximately equal to 2.71828…) as its base. It is written as "ln" and is used to determine how many times we need to multiply

$$e^y = x \Leftrightarrow \ln(x) = y$$

For example, the natural logarithm of 7.389 is about 2, because $2.71828^2 \approx 7.389$.



2.3.1 Euler's number "*e*"

The natural logarithm base, denoted by "*e*," not only features in compound interest but naturally emerges in various fields like finance, calculus and number theory. It's a go-to for modeling exponential growth and decay, playing a crucial role in advanced mathematical studies compared to the common logarithm.

Class Activity

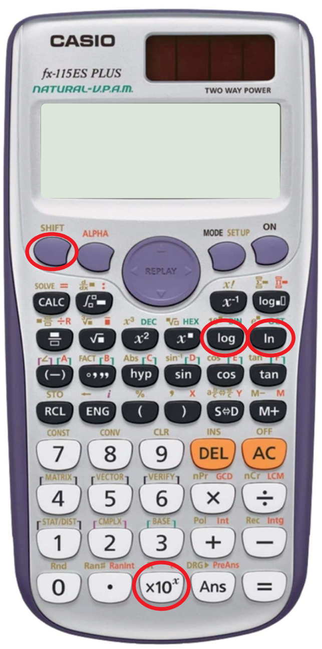
The expression below has three digits. Swap the position of two of the digits to make an expression with:

$$log_8 \frac{2}{4}$$

- The highest value.
- The lowest value.

Note

Remember, while the bases of these logarithms are different, the fundamental properties and laws of logarithms (like the product rule, quotient rule and power rule) apply to both types. The primary distinction is the base and the contexts in which they are typically used.



Relation between common log and natural log by using change of base formula, which is,

$$log_{10} x = \frac{ln\, x}{ln 10},$$

where:

- $log_{10} x$ is the common logarithm of x .
- $ln\, x$ is the natural logarithm of
- $ln 10$ is the natural logarithm of 10, which is approximately equal to 2.30259.

Therefore, $ln\, x = 2.3025 \times log_{10} x$.

Knowledge 2.4 Antilogarithm

The antilogarithm is about "undoing" the logarithm. If you take the antilog of a log, you get back to your original number.

Relationship with Exponentiation

It is inherently an operation of exponentiation. For base 10 logarithms, finding the antilog of a number is equivalent to raising 10 to the power of that number.

The antilogarithm (often called the antilog) is the inverse function of the logarithm function. This means if $y = \log_b x$, then $x = b^y$ or $x = \text{antilog}_b(y)$.

For common logarithms the antilog is the same as raising 10 to the power of y, $10^y = x$.

For natural logarithms the antilog is the same as raising e to the power of $y, e^y = x$.

How to calculate Log & Antilog on a calculator?

Common Log: To find the common logarithm, Press the **“log”** button and type [Your Number].

Natural Log: To find the natural logarithm, Press the **“ln”** button and type [Your Number].

For Common Antilog: Press **"Shift"** then **"log"** then [Your Number].

For Natural Antilog: Press **"Shift"** then **"ln"** then [Your Number].

If the *ln* and *log* functions are not available, you can use the highlighted buttons in the last row:

- Press 10^x for the common log.
- Press “e^x” for the natural log.

Skill 2.2

- Converting logarithms into exponential and vice versa.
- Applying definition of log to find the unknowns.
- Choose appropriately between common and natural logarithms based on the context of the problem.

Exercise 2.2

1. Convert the following from exponential form to logarithmic form.

i. $9^{-2}=\frac{1}{81}$ **ii.** $10^{-3} = 0.001$ **iii.** $4^{-2} = \frac{1}{16}$

iv. $3^{-4}=\frac{1}{81}$ **v.** $\left(\frac{1}{5}\right)^{2} = \frac{1}{25}$ **vi.** $\left(\frac{1}{3}\right)^{-3} = 27$

2. Convert the following from logarithmic form to exponential form.

i. $log_u v = -16$ **ii.** $log_7 7 = 1$ **iii.** $log_{\frac{1}{2}} \frac{1}{8} = 3$ **iv.** $log_3 1 = 0$

v. $log_e \frac{1}{64} = -x$ **vi.** $log_6 \frac{1}{36} = -2 log_{\frac{7}{4}} x$ **vii.** $log_5 25 = y$ **viii.** $ln\, x = -8$

3. Find the value of unknown variable

i. $log_{16} 4 = y$ **ii.** $log_2 8 = y$ **iii.** $log_7 \frac{1}{7} = y$

iv. $log_y 32 = 5$ **v.** $log_5 n = 2$ **vi.** $log_{64} 8 = \frac{x}{2}$

4. Find the value of unknown variable.

i. $e^x = 4$ **ii.** $ln x = 6$ **iii.** $ln(2x-1) = 1$ **iv.** $e^{3x+5} = 10$

v. $ln\left(e^{3-x}\right) = 8$ **vi.** $e^{e^x} = 5$ **vii.** $ln x = \frac{1}{2}$ **viii.** $e^x = 7$

5. Evaluate

i. $log_{2\sqrt{2}} 512$ **ii.** $log_3 \frac{1}{243}$ **iii.** $log_3 1 = y$ **iv.** $log_6 \frac{1}{216}$

v. log (0.01) **vi.** $log_{343} 7 log_{\frac{1}{6}} 216$ **vii.** $log_{\frac{2}{3}} \frac{8}{27}$ **viii.** $log_5 \sqrt[3]{5}$

6. Find between which two consecutive integers the logarithm lies.

i. $log_2 30$ **ii.** $log_7 9$ **iii.** $log_3 75$ **iv.** $log_{10} 7500$

Knowledge 2.5 Laws and Properties of Logarithm

The laws and properties of logarithms are essential tools in mathematics, transforming complex multiplicative relationships into simpler additive forms. They are pivotal for solving equations, understanding exponential growth and decay and facilitating advanced mathematical analysis.

(a) Power Rule

The logarithm of a number raised to an exponent equals the exponent multiplied by the logarithm of the base number. This law enables us to extract the exponent from the logarithm and use it as a multiplier.

Expression: $log_b m^n = n \times log_b m$

Let $log_b m^n = x \cdots\cdots\cdots(i)$ and $log_b m = y \cdots\cdots\cdots(ii)$

Exponential Form: $b^x = m^n$ and $b^y = m$

$b^x = m^n$

$b^x = \left(b^y\right)^n$ ∵ Replaced with b^y

$b^x = b^{yn} \Rightarrow x = yn \cdots\cdots\cdots(iii)$ ∵ By Comparing

We have,

$log_b m^n = x$ and $x = yn$ ∵ From (i) and (iii)

$log_b m^n = ny$

$log_b m^n = nlog_b m$ ∵ Replaced y with $log_b m$

Hence proved.

Let us consider the following example.

Example 2.8

Solve $log_4\left(4^5\right)$.

Solution:

$log_4\left(4^5\right) = 5log_4(4) = 5$ ∵ $log_b b = 1$

(b) Product Rule

The logarithm of a product equals the sum of the logarithms of its factors. This law enables us to break down the logarithm of a multiplication into the sum of two separate logarithms.

Expression: $\log_b(m \times n) = \log_b(m) + \log_b(n)$

Let $\log_b m = x$ and $\log_b n = y$

Exponential form: $b^x = m \cdots\cdots\cdots(i)$ and $b^y = n \cdots\cdots\cdots(ii)$

Multiplying (i) and (ii)

$$mn = b^x \times b^y \Rightarrow mn = b^{x+y}$$

Taking log with base b on both sides,

$\log_b(m \times n) = \log_b\left(b^{x+y}\right)$

$\log_b(m \times n) = (x+y)(\log_b b)$ ∵ By power law

$\log_b(m \times n) = x + y$ ∵ By identity law

$\log_b(m \times n) = \log_b m + \log_b n$ ∵ $\log_b m = x$ and $\log_b n = y$

Let us consider the following example.

Example 2.9 Solve $log_5\left(5^2 \times 5^3\right)$.

Solution:

$$log_5\left(5^2 \times 5^3\right) = log_5 5^2 + log_5 5^3 = 2log_5 5 + 3log_5 5$$

$$= 2(1) + 3(1) = 5 \qquad \because\ log_b\,\mathrm{b} = 1$$

(c) Quotient Rule

The logarithm of a quotient is the difference between the logarithms of the numerator and the denominator. This law allows us to transform the logarithm of a division into the subtraction of two logarithms.

Expression: $log_b\left(\frac{m}{n}\right) = log_b(m) - log_b(n)$

Let $log_b m = x$ and $log_b n = \mathrm{y}$

Exponential form: $b^x = m$ ·········(i) and $b^y = n$ ·········(ii)

Dividing (i) by (ii),

$$\frac{m}{n} = \frac{b^x}{b^y} \Rightarrow \frac{m}{n} = b^{x-y}$$

Taking log with base b on both sides ,

$$log_b\left(\frac{m}{n}\right) = log_b\left(b^{x-y}\right)$$

$$log_b\left(\frac{m}{n}\right) = (x-y)(log_b\, b) \quad \because \text{By power Law: } log_b m^n = n\,log_b m$$

$$log_b\left(\frac{m}{n}\right) = x - y \qquad \because \text{By identity law: } log_b b = 1$$

$$log_b\left(\frac{m}{n}\right) = log_b m - log_b n \quad \because \text{By substitution: } log_b m = x \text{ and } log_b n = y$$

Test Yourself

Express

$$log\left(\frac{75}{16}\right) - 2log\left(\frac{5}{9}\right) + log\left(\frac{32}{243}\right)$$

in terms of log 2 and log 3.

Let us consider the following example.

Example 2.10 Solve $log_3\left(\frac{27}{3}\right)$

Solution:

$$log_3\left(\frac{27}{3}\right) = log_3(27) - log_3 3 = log_3 3^3 - log_3 3$$

$$3\log_3 3 - \log_3 = 3(1) - 1 = 2$$

(d) Identity Law

The logarithm of a number to its own base is always 1. This reflects the idea that any number raised to the power of 1 is itself.

Expression: $log_b b = 1$

Let $log_b b = x \cdots\cdots\cdots (i)$

Exponential form: $b^x = b \quad \Rightarrow \quad b^x = b^1 \quad \Rightarrow \quad x = 1$

From (i) we have,

$log_b b = x = 1$

$log_b b = 1$

For example, $log_2 2 = 1$

(e) Change of Base Formula

This formula offers a method to express a logarithm in one base in terms of logarithms in another base, proving especially useful for computations involving specific bases.

Expression: $log_b a = log_b c \times log_c a = \dfrac{log_c a}{log_c b}$

From R.H.S, let $log_c a = x \cdots\cdots\cdots (i)$

Exponential Form: $c^x = a$

Taking log with base b on both sides,

$log_b a = log_b c^x$

$log_b a = x log_b c$ $\quad \because$ By power law: $log_b m^n = n log_b m$

$log_b a = log_c a . log_b c \cdots\cdots\cdots (ii)$ $\quad \because$ Replaced x with $log_c a$

Now replace a with b,

$log_b b = log_c b . log_b c$

$1 = log_c b . log_b c$

$$\frac{1}{log_c b} = log_b c \cdots\cdots\cdots (iii)$$

Replace $log_b c$ with $\dfrac{1}{log_c b}$ in (ii), we get

$log_b a = \dfrac{log_c a}{log_c b}$ or we can also say that

$log_c a = log_b a \times log_c b$

Test Yourself

Prove that

$log_{12} 1728 \times log_9 6561 = 7$

Teacher's Guidelines

Use measuring cups to teach logarithms, highlighting how doubling or halving recipes reflect logarithmic scaling. This visual method, showing exponential growth or division, makes the abstract concept of logarithms more tangible and relatable by linking it to everyday objects.

Example 2.11 Express $log_5 130$ in base 10.

Solution:

Using change of base formula,

$$log_5 130 = \frac{log_{10} 130}{log_{10} 5}$$

(f) Inverse Property

Exponentiation and logarithms are inverse operations. Raising a base to its logarithm (with the same base) yields the original number, reinforcing the concept that logarithms and exponentiation are operations that undo each other.

Expression: $b^{log_b x} = x$ and $log_b b^x = x$

For example, $10^{log_{10} 100} = 100$

Skill 2.3

- Utilizing the laws of logarithms to combine or separate logarithmic terms.
- Utilizing the laws of logarithms to evaluate logarithmic equations.

Exercise 2.3

1. Expand the following using laws and properties of logarithm.

   i. $log(A \times B \times C)$ ii. $log\frac{15.2 \times 30.5}{81.8}$ iii. $log\sqrt[3]{\frac{7}{15}}$ iv. $log_a \frac{y}{\sqrt[3]{x}}$

   v. $log_5\left(\frac{x\sqrt{x}}{125}\right)$ v. $log(x^2y^3)$ vii. $log_3 \sqrt[4]{m^5n^2}$ viii. $log_3 \frac{xy^3}{a^3b^2c}$

2. Write the following in single simplified form.

   i. $log_2 7 + log_2 4$ ii. $log 25 + log 4$ iii. $log_4 2x + log_4 4x^2$

   iv. $log_3 24 - log_3 8$ v. $log_4 x^9 - log_4 x^2$ vi. $log_3 4 + log_3 y + \frac{1}{2}log_3 49$

   vii. $\frac{1}{3}(log_5 8 + log_5 27) - log 3$ viii. $2 log 6 - \frac{1}{4}log 16 + log 3$

3. Find x if

   i. $log 2x + 1 = 2$ ii. $2log_b 4 + log_b 5 - log_b 10 = log_b x$ iii. $log_b 30 - log_b 5^2 = log_b x$

   iv. $log_b 8 - log_b x^2 = log_b x$ v. $log_b(x+2) - log_b 4 = log_b 3x$ vi. $log_b(x-1) + log_b 3 = log_b x$

   vii. $log_x 81 = 4$

4. If $log 8 = x$ and $log 3 = y$, express the following in terms of x and y.

   i. $log 24$ ii. $log\frac{9}{8}$ iii. $log 720$

5. Evaluate each of the following logarithms.

   i. $log_{25} 5$ ii. $log_{64} 2$ iii. $log_{\frac{1}{4}} 8$

6. Express each of the following as simply as possible in terms of logarithms to base b for the given value of b.

i. $log_5 7,\ b=2$ ii. $log_{\frac{1}{3}} 8,\ b=10$ iii. $\frac{1}{log_6 31},\ b=5$

7. If $log\,2 = 0.3010$, $log\,3 = 0.4771$ and $log\,5 = 0.6990$ then find the values of the following

a. $log\,30$ b. $log\sqrt{4\frac{4}{5}}$ c. $log\frac{20}{3}$ d. $log(5^{2.5}\times\sqrt[3]{3})$

Knowledge 2.6 Real World Applications of Logarithm

Student Learning Outcomes

- Apply laws of logarithm to real life situations such as growth and decay, loudness of sound.

Logarithms are used for extracting powers or roots and are, in fact, corollaries to the theory of indices. This is why they can transform very complex problems into simpler ones. They help us manage and visualize both extremely large and tiny numbers, ensuring that the results remain consistent.

Teaching Guidelines

Use measuring cups to teach logarithms, highlighting how doubling or halving recipes reflect logarithmic scaling. This visual method, showing exponential growth or division, makes the abstract concept of logarithms more tangible and relatable by linking it to everyday objects.

Understanding Scales: Linear vs. Logarithmic

Linear Scaling: Features equal intervals, such as 1, 2, 3,… or 10, 20, 30,… resembling an arithmetic sequence. Each step adds a constant value.

Logarithmic Scaling: Involves multiplicative steps, for example, 1, 10, 100, 1000,… following a geometric progression. This scaling is exemplified by the number line discussed previously in section 2.2, where each increase is by a factor, not a sum.

The Usability of Logarithmic Scales

Linear scales can make it challenging to discern the growth of human population from thousands to billions, as they compress the vast differences into uniform intervals. In contrast, logarithmic scales adeptly illustrate and allow for easy comparison of both early and recent population growth within a single graph, as demonstrated below.

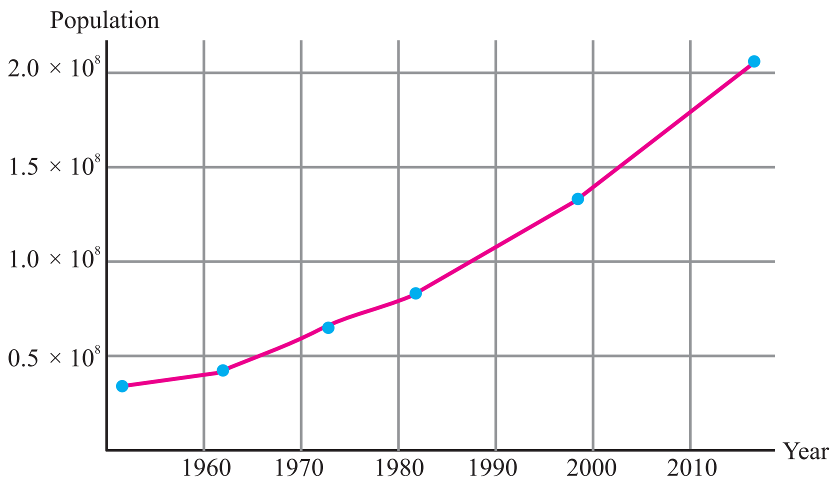
In order to have a better understanding of logarithmic scale,w plot the data mentioned in the table 2.1 on a cartesian plane.

Table 2.1

X[Years]	Y [Population]
1951	33,740,167
1961	42,880,378
1972	65,309,340
1981	84,254,644
1998	132,352,279
2017	207,774,520

Pakistan Population Census Trends: 1961-2017

The y-axis values double proportionally, while the x-axis uses a linear scale incrementing by 10.



Interesting Information

Logarithmic scales make it easier to see big differences in data and are used in measuring sound (decibels), earthquake strength (Richter scale), acidity (pH scale) and light intensity (stellar magnitudes).

Exponential Growth and Decay

Exponential growth occurs when something multiplies at a rate that depends on how much of it is already there. This leads to rapid increases, like when one flower produces three, then nine flowers.

On the other hand, exponential decay is when something decreases faster when there's more of it, gradually slowing down as it gets smaller. It's a bit like eating a chocolate bar and halving what's left each day.

Exponential Growth in Pyramid Schemes: The Recruitment Model

Logarithm in Complex Numerical Calculations

Logarithms are indispensable in mathematics and practical applications, aiding in complex calculations and understanding growth, decay and sound intensity in fields like finance, biology and physics.

Riddle

You'll find me between an ant's weight and a mountain's height. In linear terms, I'm vast, but in my terms, I'm just right. What am I?

Example 2.12 Solve $\sqrt[3]{\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}}$

Solution:

Let $y=\sqrt[3]{\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}} \Rightarrow y=\left(\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}\right)^{\frac{1}{3}}$

Taking log on both sides

$$\log y=\log\left(\dfrac{0.821\times(45.23)^4}{(5.79)^3\times 0.942}\right)^{\frac{1}{3}}$$

$$log\ y = \frac{1}{3} log \frac{0.821 \times (45.23)^4}{(5.79)^3 \times 0.942}$$

$$\log y = \frac{1}{3}\left[\log 0.821 + \log(45.23)^4 - \log(5.79)^3 - \log 0.942\right]$$

$$\log y = \frac{1}{3}[\log 0.821 + 4\log 45.23 - 3\log 5.79 - \log 0.942]$$

$$\log y = \frac{1}{3}\left[-0.0856 + 4(1.6554) - 3(0.7626) - (-0.0259)\right]$$

$$\log y = \frac{1}{3}[4 - 0.0856 + 6.6216 - 2.2878 + 0.0259]$$

$$\log y = \frac{1}{3}[4.2741] \Rightarrow \log y = 1.4274$$

To eliminate log, take antilog on both sides

$$\text{antilog}(\log y) = \text{antilog}(1.4274) = 26.5888$$

Therefore, $y = 26.5888$

Example 2.13

The intensity I_1 of a whisper is 1×10^{-12} watts per square meter (W/m^2). The intensity I_2 of normal conversation is 1×10^{-6}. Calculate the difference in sound intensity levels (in decibels) between a whisper and normal conversation.

Solution:

The formula to calculate the sound intensity level L in decibels (dB) is given by:

$$L = 10 \times log\left(\frac{I}{I_o}\right),$$

where,

- I is the intensity of the sound.
- I_0 is the reference intensity, which is 1×10^{-12} W/m^2 (it is always constant).

Sound intensity level for whispers:

$$L_1 = 10 \times \log\left(\frac{1 \times 10^{-12}}{1 \times 10^{-12}}\right) \Rightarrow L_1 = 10 \times \log 10^0$$

$L_2 = 10 \times 0 \Rightarrow L_1 = 0$ dB

Sound intensity level for normal conversation:

$$L_2 = 10 \times \log\left(\frac{1 \times 10^{-6}}{1 \times 10^{-12}}\right) \Rightarrow L_2 = 10 \times \log 10^6$$

$$L_2 = 10 \times 6 \Rightarrow L_2 = 60 \text{ dB}$$

$$L = L_2 - L_1 = (60 - 0) = 60 \text{ dB}$$

Example 2.14

Starting with 500 users, a startup experiences continuous growth at a rate of 8% per year. To determine the number of years it takes for the company to reach 3000 users, use the formula $N(t) = N_0 \times e^{rt}$

where $N(t)$ is the number of users after time t, N_0 is the initial number of users, r is the growth rate and t is the time in years

Solution:

Given that, $N_0 = 500$ users, r = 0.08 (8% expressed as a decimal), $N(t) = 3000$, t=?

Now insert the values into the formula,

$$N(t) = N_0 \times e^{rt}$$

$$3000 = 500e^{0.08 \times t}$$

To isolate "t" in the equation, we apply the natural logarithm (ln) to both sides, utilizing the power rule of ln, which allows us to move the exponent to the coefficient position for easier manipulation.

$$ln\, 3000 = ln\,(500e^{0.08t})$$

$$ln\, 3000 = ln\, 500 + 0.08t(ln\, e)$$

$$8.0063 = 6.2146 + 0.08t(1) \Rightarrow 8.0063 - 6.2146 = 0.08t$$

$$1.7917 = 0.08t$$

$$\frac{1.7917}{0.08} = t \Rightarrow t = 22.39 \approx 22.4 \text{ years}$$

The company is projected to reach 3,000 members from an initial count of 500 in approximately 22 years and 3 months.

Skill 2.4

Use logarithmic functions to analyze real-life situations such as population growth, radioactive decay, and sound intensity levels.

Exercise 2.4

1. Use logarithm to evaluate the following expressions.

i. $\sqrt[4]{2.145}\times\dfrac{\sqrt{15.236}}{6000}$ **ii.** $\dfrac{183\times\sqrt[3]{2}}{0.2356\times\sqrt[5]{2578}}$ **iii.** $\dfrac{(438)^3\times\sqrt{0.056}}{(388)^4}$

2. If an amount of PKR 50,000 is invested in a bank account that offers compound interest at a rate of 4% annually, compounded quarterly, how many years will it take for the amount to grow to PKR 70,000? (Hint: $A=P\left(1+\dfrac{r}{n}\right)^{nt}$, where A is the future value, P is the principal amount, r is the annual interest rate, n is the number of times interest is compounded per year and t is the time in years.)

3. The value, PKR V, of an investment after t years is given by the formula $V=Ae^{0.03t}$, where PKR A is the initial investment.

i. How much will an investment of PKR 4000 be worth after 3 years?

ii. To the nearest year, how long will you need to keep an investment for it to double in value?

4. The value of a new car depreciates at a rate of 20% per year. If the car is initially worth PKR 25,000,00 how many years will it take for its value to drop below PKR 10,000,00?

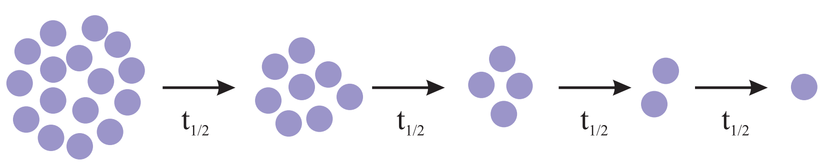
(Hint: $V(t)=V_0\times(1-r)^t$, where $V(t)$ is the value after time t, V_0 is the initial value and r is the annual depreciation rate.)

5. A town has a population of 50,000 people. The population is increasing at an annual rate of 6%. If this growth rate continues, in how many years will the town's population reaches 100,000?

(Hint: $P(t)=P_0\times e^{rt}$, where $P(t)$ is the population after time t, P_0 is the initial population, r is the growth rate, and e is the base of the natural logarithm.)

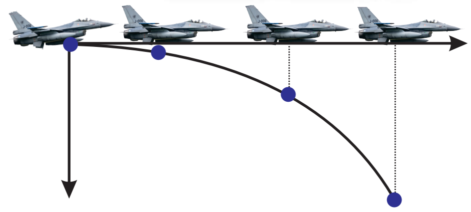
6. A radioactive element has a half-life (time in which half of the radioactive sample is left) of 5,000 years. If a sample initially contains 80 grams of the element, how long will it take for only 10 grams to remain?

(Hint: $A(t)=A_0\times\left(\dfrac{1}{2}\right)^{\frac{T}{t}}$, where $A(t)$ is the amount after time t, A_0 is the initial amount and T is the half-life.)



7. Considering that the intensity of a quiet room (I_1) is 1×10^{-1} watts per square meter $(\mathrm{W/m^2})$ and the intensity of busy street traffic (I_2) is $1\times10^{-5}\ W/m^2$, can you determine the difference in sound levels in decibels (dB) between these two environments using logarithmic calculations?

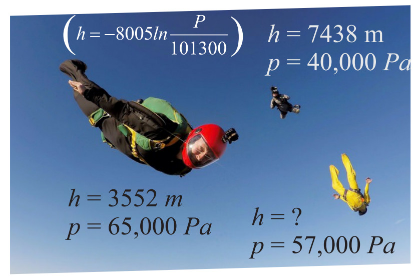
8. Considering that prolonged exposure to sounds above 85 decibels can cause hearing damage or loss and considering that a gunshot from a .22 rimfire rifle has an intensity of about $I = (2.5 \times 10^{13})$, should you follow the rules and wear ear protection when practicing at the rifle range?



9. The path of a projectile launched from an aircraft is given by the equation $h = 5000 - e^{0.2t}$, where h is the height in meters and t is the time in seconds.

i. From what height was the projectile launched?

ii. The projectile is aimed at a target at ground level. How long does it take to reach the target?



10. Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude h (in meters) above sea level is related to the air pressure P (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals? $\left(h = -8005 ln \frac{P}{101300}\right)$

11. The energy magnitude M of an earthquake can be modeled by $M = \frac{2}{3} log\, E - 99$, where E is the amount of energy released. What is the magnitude of an earthquake with an energy release of 7.079×10^{26}? Round your answer to the nearest whole number.

Review Exercise 2

1. Identify True or False

i. Converting a number from standard form to scientific notation always results in a smaller numerical value.

ii. The expression 3.6×10^4 can be written as 36×10^3 in scientific notation.

iii. The logarithm of a negative number is always undefined.

iv. Logarithms are only defined for positive base values.

v. The natural logarithm (ln) is always larger than the common logarithm (log_{10}) for the same input value.

vi. The natural logarithm (ln) has special significance in calculus due to its role in solving exponential growth and decay problems.

vii. $log\left(\frac{x}{y^3}\right) = logx - 3logy$

viii. $log(a-b) = loga - logb$

ix. $-ln\left(\frac{1}{x}\right) = lnx$

x. $ln_{\sqrt{x}} x^k = 2k$

2. Four every question, there are four options, choose the right one.

i. If $a^x = n$, then

(a) $a = log_x n$ (b) $x = log_n$ a (c) $x = log_a n$ (d) $a = log_n x$

ii. $log_e 10 =$

(a) 2.3026 (b) 0.4343 (c) e^{10} (d) 10

iii. If $log_2 x = 5$ then x is:

(a) 25 (b) 32 (c) 10 (d) 2^{5x}

iv. If $log\,27 = 1.431$, then the value of $log\,9$ is:

(a) 0.934 (b) 0.945 (c) 0.954 (d) 0.958

v. If $log\frac{a}{b} + log\frac{b}{a} = log(a+b)$, then:

(a) $a+b=1$ (b) $a-b=1$ (c) $a=b$ (d) $a^2-b^2=1$

vi. If $log_{10}70 = a$, then $log_{10}\left(\frac{1}{70}\right)$ is equal to

(a) $-(1+a)$ (b) $(1+a)^{-1}$ (c) $\frac{a}{10}$ (d) $-a$

vii. $log\,4 + log\,25$=?

(a) 2 (b) 3 (c) 4 (d) 5

viii. $log_2 7$ is

(a) An integer (b) A rational number (c) An irrational number (d) A prime number

ix. 0.36^2 in standard form is

(a) 0.01296×10^1 (b) 1.296×10^{-1} (c) 0.1296×10^0 (d) 12.96×10^{-2}

x. What is the value of x in the exponential equation $9 + e^{2x-4} = 1$?

(a) 2 (b) 3 (c) 4 (d) 5

xi. $log_3 \sqrt[9]{81} =$

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $\frac{4}{9}$

xii. $log_b a \times log_c b$ can be written as

(a) $log_c a$ (b) $log_a c$ (c) $log_a b$ (d) $log_b c$

xiii. $log_y x$ will be equal to

(a) $\frac{log_z x}{log_y z}$ (b) $\frac{log_x z}{log_y z}$ (c) $\frac{log_z x}{log_z y}$ (d) $\frac{log_z y}{log_z x}$

xiv. Simplify $10^{12} \div 10^5 \div 10^3$

(a) 10^{20} (b) 10^{12} (c) 10^{10} (d) 10^4

xv. If $log_b x = 4$ and $log_b y = 2$, what is $log_b(x \times y^3)$?

(a) 5 (b) 7 (c) 9 (d) 10

3. Write the following in standard form.

i. 0.000094 **ii.** 865000000 **iii.** 729.89×10^3

4. Write the following in ordinary form.

i. 1.6×10^{-3} **ii.** 4.8×10^{-2} **iii.** 5.12×10^{-3}

5. Solve for x.

i. $log_{625} 5 = \frac{1}{4}x$ **ii.** $log_{64} x = -\frac{2}{3}$ **iii.** $log_{36} x = -\frac{1}{2}$ **iv.** $log_x 16 = 2$

6. Express in terms of x, y and z if $log_7 2 = x$, $log_7 3 = y$ and $log_7 5 = z$.

i. $log_7 60$ **ii.** $log_7 \frac{50}{27}$ **iii.** $log_7 \frac{15}{2}$ **iv.** $log 10.5$

7. Simplify

i. $log_b x^2 + log_b x^3 - log_b x^4$ **ii.** $log_k \frac{a}{b} + log_k \frac{b}{a}$ **iii.** $log_b(x^2 - a^2) - log_b(x - a)$ if $x > a$

8. Prove the following statements.

i. $log_{\sqrt{b}} x = 2 log_b x$ **ii.** $log_{\frac{1}{\sqrt{b}}} \sqrt{x} = -log_b x$

9. Solve the following logarithmic equations.

i. $log x + log(x - 3) = 1$ **ii.** $log(x - 2) + log(x + 1) = 2$

iii. $2 log x = log 2 + log(3x - 4)$ **iv.** $log x + log(x - 1) = log 4x$

10. A radioactive substance is decaying such that its mass, m grams, at a time t years after initial observation is given by $m = 240e^{kt}$ where k is a constant. Given that when $t = 180$ and $m = 160$,

i. find the value of k. **ii.** calculate the time it takes for the mass of the substance to be halved.

11. A quantity N is increasing such that at time t, $N = 20e^{0.04t}$.

i. Find the value of N when $t = 15$. **ii.** Find, in terms of the constant k, expressions for the value of t when $N = k$.

12. A quantity N is decreasing such that at time t, $N = N_0 e^{kt}$. Given that at time $t = 10$, $N = 300$ and that at time $t = 20$, $N = 225$, find

i. the values of the constants N_0 and k. **ii.** the value of t when $N = 150$.

Review Check List

1. ▢ **Scientific Notation**
A method to represent numbers as a significant digit multiplied by 10^n. Useful for managing very large or small numbers.

a. ▢ **Standard Form**
Describes how any number, X, can be expressed as $A\times10^n$. Details conversion based on shifting the decimal.

b. ▢ **Ordinary Form**
A method to revert from scientific notation by removing the power of ten.

c. ▢ **Arithmetic Operations on Scientific Notations**
Rules for multiplication, division, addition, and subtraction using numbers in scientific notation.

2. ▢ **Logarithm**
"The logarithm of a positive real number "a" with respect to base "b", a positive real number not equal to 1 is the exponent by which b must be raised to yield "a". For instance, $\log_{10}100=2$.
Relationship between exponents and logarithms.
$b^x = a \Leftrightarrow \log_b a = x$, where $b > 0$, $b \neq 1$ & $a > 0$.

a. ▢ **Understanding via number line and graph**

b. ▢ **Points to Remember**
Logarithm of a Positive Number is Negative
Negative results in logarithms lead to expressions like $a^{-n} = \frac{1}{a^n}$.
Logarithm of a Negative Number
Logarithm of a negative number isn't possible in real numbers.
Concept of Negative Base
Logarithms can't have negative bases in real numbers context.
Logarithm of One
Logarithms with an argument of 1 are always 0 regardless of the base.
Undefined Logarithm
Logarithms are undefined for arguments of zero or negative numbers in real numbers context.

3. ▢ **Natural Logarithm**
Logarithm with base 'e', where 'e' is Euler's number (~2.71828).

a. ▢ **Euler's Number "e"**
Explains 'e' as a constant arising in various mathematical contexts, particularly in growth and decay models.

4. ▢ **Antilogarithm**
if $y = \log_b(x)$, then $x = b^y$ or $x = \text{antilog}_b(y)$

5. ▢ **Laws and Properties of Logarithm**
a. Product Law: $\log_b(m \times n) = \log_b m + \log_b n$
b. Quotient Law: $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$
c. Power Law: $\log_b m^n = n \log_b m$
d. Change of Base Formula: $\log_b a = \log_c b \times \log_c a$ for any base c.
e. Identity Property: $\log_b b = 1$
f. Inverse Property: $b^{\log_b x} = x$ & $\log_b b^x = x$

6. ▢ **Real World Applications & Uses of Logarithm**
Logarithms, acting as the inverse of exponentials, simplify complex calculations: they turn multiplication into addition, division into subtraction, and exponentiation into multiplication, making large and small number operations manageable
a. Logarithmic Scaling
Linear Scales: Uniform steps, e.g., 1, 2, 3 or 10, 20, 30, comparable to an arithmetic sequence.
Logarithmic Scales: Multiplicative progression (e.g., 1, 10, 100) and geometric in nature.
b. Clarity in Visualization: Log scales depict datasets with vast variations without losing details.
c. Mathematical Ease: Makes operations simpler
d. In Sound Engineering: Decibel (dB) scale helps in representing sound intensities.

Mathematics

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